INTRODUCTION TO

RANDOM PLANAR GEOMETRY

LECTURE I - DISCRETE PLANE TREES

① Framework

Convention : $\mathbb{N} = \{1, 2, \dots\}$

$\mathbb{N}^0 = \{\emptyset\}$

Introduce the set

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

An element $u \in \mathcal{U}$ can be written

$$u = (u_1 \dots u_n)$$

[We shall often write $u = u_1 \dots u_n$]

We call $|u| = n$ the generation of u.

CONCATENATION RULE :

for $u = u_1 \dots u_n$ & $v = v_1 \dots v_m$, define

$$uv := u_1 \dots u_n v_1 \dots v_m .$$

NOTE : $u\emptyset = \emptyset u = u$.

Definition

A *plane tree* τ is a finite subset of $\mathcal{U}$ such that:

(i) $\phi \in \tau$

(ii) if $u = u_1 \dots u_n \in \tau$ $(n \geq 2)$, then
$$\overleftarrow{u} = u_1 \dots u_{n-1} \in \tau$$

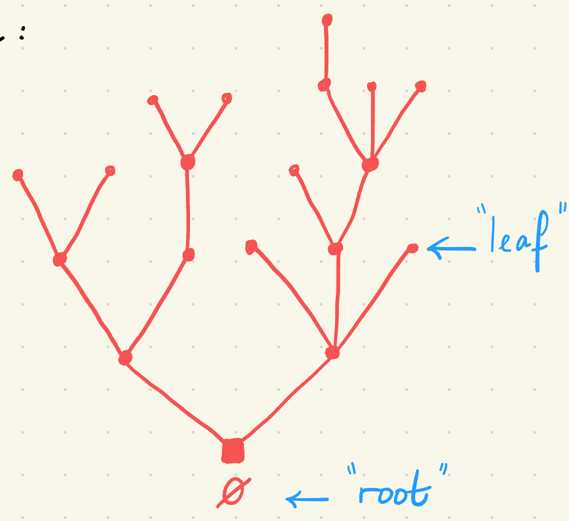
(iii) for $u \in \tau$, there exists an integer $k_u \geq 0$ s.t. for $j \in \mathbb{N}$,
$$uj \in \tau \iff 1 \leq j \leq k_u .$$

VIEWPOINT. It is useful to see each node $u \in \tau$ as an individual, where:

- $u \in \tau$ may have children $uj \in \tau$, $1 \leq j \leq k_u$

[k_u = number of children]

- $\varnothing$ is the initial ancestor (generation 0)



NOTATION.

We denote by $\mathbb{T}$ the set of all plane trees.
The **size** $|\tau|$ of a tree $\tau \in \mathbb{T}$ is its number of edges (so $|\tau| = \#\tau - 1$). Let

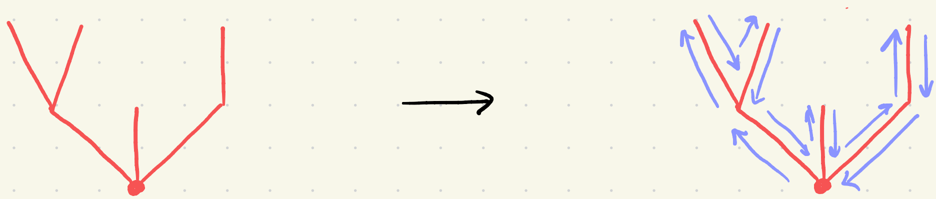
$$\mathbb{T}_k := \{\tau \in \mathbb{T}, \quad |\tau| = k\} \qquad (k \geq 0).$$

Fact. $$\# \mathbb{T}_k = Cat_k := \frac{1}{k+1}\binom{2k}{k}$$

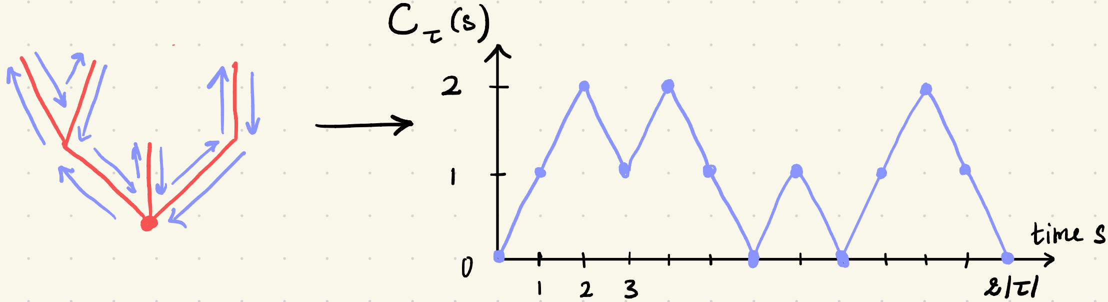
[The proof is a simple recursion which is omitted]

CONTOUR FUNCTIONS.

Take a tree τ — Draw contour of τ (from left to right)



This gives a sequence of up/down ↗ ↘ arrows



C_τ = contour function of τ.

$C_\tau(s)$ = height of individual at time s in τ.

C_τ is a Dyck path of length $2|\tau|$: a Dyck path of length k is a sequence $(x_0, \dots, x_{2k})$ of nonnegative integers with $x_0 = x_{2k} = 0$ and $|x_i - x_{i-1}| = 1$ for all $i \in \{1, \dots, 2k\}$.

Proposition.

Let $k \geq 0$. The mapping $\tau \mapsto C_\tau$ is a bijection from $\mathbb{T}_k$ onto the set of Dyck paths of length $2k$.

PROOF BY PICTURE.

In order to get τ out of C_τ, stick some "glue" underneath the path C_τ and fold C_τ.

② Bienaymé-Galton-Watson trees

Let μ a probability measure on $\mathbb{Z}_+$ s.t.

$$\sum_{k=0}^{\infty} k\,\mu(k) \leq 1$$

(μ is said to be subcritical)

We assume that μ is non-degenerate, i.e $\mu(1) < 1$.

Interpretation: μ is the offspring distribution, i.e. the distribution on the number of children.

Subcriticality means that each individual has less than 1 child on average.

Let $(K_u, u \in \mathcal{U})$ iid law μ

and define

$$T = T_\mu := \{ u \in \mathcal{U} : \quad \forall j \leq |u| \quad u_j \leq K_{u_1, \dots, u_{j-1}} \}$$

Every $u \in T$ now has a (random) number of children $k_u(T_\mu) = K_u \sim \mu$.

Proposition.

1) T is a tree a.s.

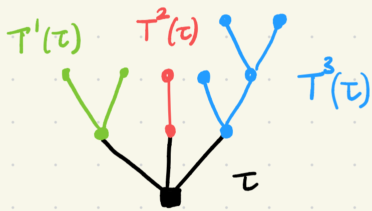
2) Let $Z_n := \#\{u \in T, |u| = n\}$, $n \geq 0$.
Then $(Z_n, n \geq 0)$ is a Galton-Watson process with offspring distribution μ.

The tree T is called a Bienaymé-Galton-Watson (BGW) tree with offspring distribution μ.
We shall write $\mathbb{P}_\mu$ for the law of T_μ (this is a probability measure on $\mathbb{T}$).

For a tree $\tau \in \mathbb{T}$, and $j \in \{1, \dots, k_\varnothing(\tau)\}$, one may consider the shifted tree

$$T^j(\tau) = \{u \in \mathcal{U} : \; ju \in \tau\}.$$

$T^j(\tau)$ is a tree.



We simply write T^j instead of $T^j(T_\mu)$ for the shifts of T_μ.

T_μ enjoys the following fundamental property.

Proposition. [Branching property of BGW trees]

Let $j \geq 1$ with $\mu(j) > 0$.

Under $\mathbb{P}_\mu(\cdot \mid K_\varnothing = j)$, the shifted trees $T^1, \dots, T^j$ are iid. samples of $\mathbb{P}_\mu$.

Proposition.

For $\tau \in \mathbb{T}$,

$$\mathbb{P}_\mu(\tau) = \prod_{u \in \tau} \mu(k_u(\tau))$$

PROOF. By definition of T,

$$T = \tau \iff \forall u \in \tau \quad K_u = k_u(\tau)$$

Hence

$$\mathbb{P}_\mu(\tau) = \mathbb{P}\Big(\bigcap_{u \in \tau} \{K_u = k_u(\tau)\}\Big)$$

$$\overset{(\perp\!\!\!\perp)}{=} \prod_{u \in \tau} \mathbb{P}(K_u = k_u(\tau))$$

$$= \prod_{u \in \tau} \mu(k_u(\tau)). \qquad \square$$

③ An important example: the geometric case

We restrict to a particularly nice $\mu = \mu_0$:

$$\mu_0(k) = 2^{-k-1}, \qquad k \in \mathbb{Z}_+.$$

In this case,

- $\displaystyle\sum_{k=0}^{\infty} k\,\mu_0(k) = 1$ [criticality]

- $\mathbb{P}_{\mu_0}(\tau) = 2^{-2|\tau|-1}$ depends only on $|\tau|$

As a consequence, for $k \geq 0$, $\mathbb{P}_{\mu_0}(\cdot \mid |\tau| = k)$ is just the uniform probability measure on $\mathbb{T}_k$.

The reason why μ_0 is nicer is the following connection to simple random walk. Let $(S_n, n \geq 0)$ a SRW started from $S_0 = 0$. Define

$$\sigma = \inf\{n \geq 0 : S_n = -1\}.$$

Recall that $\sigma < \infty$ a.s.

A SRW excursion is the (law of the) random finite path $(S_0, S_1, \dots S_{\sigma-1})$.

In particular, a SRW excursion remains nonnegative, starts and ends at 0.

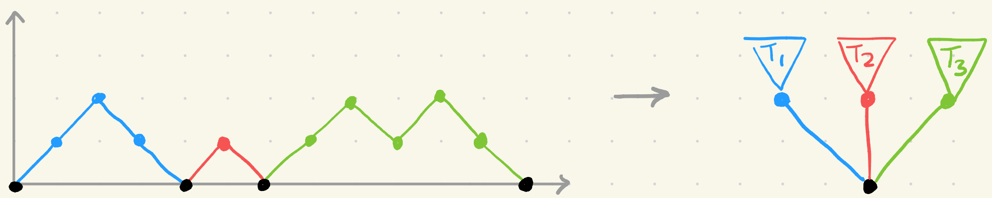
Proposition.

The Dyck path $C_{T_{\mu_0}}$ of a BGW(μ_0)-tree is a SRW excursion.

PROOF.

We prove that the tree T encoded by the Dyck path $(S_0, \dots, S_{\sigma-1})$ is indeed a BGW(μ_0)-tree.

We reveal the children of $\varnothing$ in T as follows.



"Subtrees correspond to sub-excursions"

More precisely, we introduce the random times

$$U_1 := \inf\{n \geq 0 : S_n = 1\}$$

$$V_1 := \inf\{n \geq 0 : S_n = 0\}$$

and recursively for $j \geq 1$

$$U_{j+1} := \inf\{n \geq V_j : S_n = 1\} \quad \& \quad V_{j+1} := \inf\{n \geq U_{j+1} : S_n = 0\}$$

Let $K = \sup\{ j \geq 1, \ U_j \leq \sigma \}$

K is the "number of sub-excursions" in the above picture. The construction of the mapping $C_\tau \longmapsto \tau$ entails that $K = k_\emptyset(\mathcal{T})$.

On the other hand, it also implies that the shifted trees $\mathcal{T}^1(\mathcal{T}), \ldots, \mathcal{T}^K(\mathcal{T})$ are the trees encoded by the Dyck paths

$$c_i(n) = S_{(U_i+n)\wedge(V_i-1)} - 1, \qquad 0 \leq n \leq V_i - U_i - 1$$

for $1 \leq i \leq K$.

- By the strong Markov property, given $K = k$ the trees $\mathcal{T}^1(\mathcal{T}) \ldots \mathcal{T}^k(\mathcal{T})$ are independent.
- Furthermore, for $k \geq 1$,

$$\mathbb{P}(K \geq k) = \mathbb{P}(U_1 \leq \sigma, \ldots, U_k \leq \sigma)$$

$$\overset{\substack{\text{Markov prop. at } V_1 \\ \downarrow}}{=} \mathbb{P}(U_1 \leq \sigma) \cdot \mathbb{P}(U_1 \leq \sigma, \ldots, U_{k-1} \leq \sigma)$$

$$= \mathbb{P}(U_1 \leq \sigma) \cdot \mathbb{P}(K \geq k-1)$$

By recursion, this proves that for all $k \geq 1$,

$$\mathbb{P}(K \geq k) = \mathbb{P}(U_1 \leq \sigma)^k$$

Note that $\mathbb{P}(U_1 > \sigma) = \mathbb{P}(S_1 = -1) = \frac{1}{2}$.

Therefore $\mathbb{P}(K \geq k) = 2^{-k}$ and this shows that K is μ_0-distributed.

These two points ensure that T is a $BGW(\mu_0)$ tree.

$\square$

LECTURE 2 - CRASH COURSE ON EXCURSION THEORY

① Poisson point processes

Let $(E, \mathcal{E})$ a measurable space.

A point measure on E is a measure of the form $\sum_{i=1}^{n} \delta_{e_i}$, $n \in \mathbb{N} \cup \{\infty\}$, $e_i \in E$.

The set of point measures on E is denoted $M_p(E)$.

It has a natural σ-field generated by the mappings $\nu \in M_p(E) \mapsto \nu(A)$, $A \in \mathcal{E}$.

Let μ a σ-finite measure on $(E, \mathcal{E})$.

Definition

A Poisson random measure on E with intensity μ is a rv $\mathcal{N}$ in $M_p(E)$ s.t.:

(i) For all $A \in \mathcal{E}$, $\mathcal{N}(A) \sim \mathcal{P}(\mu(A))$

[when $\mu(A) = +\infty$, $\mathcal{N}(A) = +\infty$ a.s.]

(ii) For any disjoint sets $A_1, \dots, A_m \in \mathcal{E}$, $\mathcal{N}(A_1), \dots, \mathcal{N}(A_m)$ are independent.

EXAMPLE.

Let $(N_t, t \geq 0)$ a Poisson process with parameter $\lambda > 0$. Then the process $\mathcal{N}$ defined as

$$\mathcal{N}([0,t]) = N_t, \qquad t \geq 0,$$

gives a Poisson random measure on $E = \mathbb{R}_+$ with intensity $\lambda \cdot \text{Leb}$.

Theorem

Poisson random measures exist.

Let n a σ-finite measure on $(E, \mathcal{E})$.

Definition

A Poisson point process (PPP) on E with intensity measure n is a Poisson random measure on $\mathbb{R}_+ \times E$ with intensity $\text{Leb} \otimes n$.

EXAMPLE.

Let X be a Lévy process with Lévy measure Λ

The process

$$\mathcal{N}([0,t]\times A) := \#\{0 \le s \le t : \quad X(s) - X(s^-) \in A\} \qquad \begin{matrix} t \geq 0 \\ A \in \mathcal{B}(\mathbb{R}) \end{matrix}$$

defines a PPP with intensity measure Λ.

Proposition [Characterising PPP]

Let $\mathcal{N}$ an $M_p(\mathbb{R}_+ \times E)$-valued rv.

For all $t \geq 0$ and $A \in \mathcal{E}$, let

$$N_t(A) = \mathcal{N}([0,t] \times A).$$

Then $\mathcal{N}$ is a PPP(n) iff

(i) for all $A \in \mathcal{E}$ with $n(A) < \infty$,

$(N_t(A), t \geq 0)$ is a Poisson process with parameter $n(A)$.

(ii) for all disjoint $A_1 \dots A_m \in \mathcal{E}$ with $n(A_j) < \infty$,

the processes $(N_t(A_1))_{t \geq 0}, \dots, (N_t(A_m))$ are independent.

② All you need to know about PPP's

Let $\mathcal{N}$ a PPP(n).

The following two formulas are fundamental.

Theorem

For all nonnegative measurable function f on E:

(i) [Campbell's formula:]

$$\mathbb{E}\left[\int_{\mathbb{R}_+}\int_E f(t,x)\,\mathcal{N}(dt,dx)\right] = \int_{\mathbb{R}_+}\int_E f(t,x)\,dt\,n(dx).$$

(ii) [Exponential formula:]

$$\mathbb{E}\left[\exp\left(-\int_{\mathbb{R}_+}\int_E f(t,x)\,\mathcal{N}(dt,dx)\right)\right]$$

$$= \exp\left(-\int_{\mathbb{R}_+}\int_E \left(1-e^{-f(t,x)}\right) dt\,n(dx)\right)$$

It is also often useful to relate "properties under n" and "properties under $\mathbb{P}$": this is the content of the following theorem.

Technical point: we need to assume that $(E,\mathcal{E})$ is a metric space with its Borel σ-field.

Theorem [Distribution of first point]

Let $\mathcal{N}$ a $PPP(n)$ on E, and $A \in \mathcal{E}$ s.t. $0 < n(A) < \infty$.

Define $\sigma_A := \inf\{t \geq 0 : N_t(A) \geq 1\}$.

There exists an E-valued rv e_{σ_A} s.t.

$$\forall B \in \mathcal{E} \qquad \mathcal{N}(\{\sigma_A\} \times B) = \mathbb{1}_{e_{\sigma_A} \in B}.$$

The law of e_{σ_A} is given by

$$\mathbb{P}(e_{\sigma_A} \in B) = \frac{n(A \cap B)}{n(A)}, \quad B \in \mathcal{E}.$$

Moreover, $\sigma_A \perp\!\!\!\perp e_{\sigma_A}$.

PROOF. Let us just prove the formula for the law of e_{σ_A}. Let $B \subset A$.

In terms of $\mathcal{N}$, the event $\{e_{\sigma_A} \in B\}$ is the event that $N_t(B)$ jumps before $N_t(A \setminus B)$.

Let $T_B = 1^{st}$ jump time of $N_t(B)$

$T_{A \setminus B} = 1^{st}$ jump time of $N_t(A \setminus B)$

Then T_B and $T_{A \setminus B}$ are exponential rv with respective parameters $n(B)$ and $n(A \setminus B)$ (since

$N_t(B)$ and $N_t(A\setminus B)$ are Poisson processes with those parameters). Moreover, B and $A\setminus B$ being disjoint, T_B and $T_{A\setminus B}$ are independent.
Therefore

$$\mathbb{P}(e_{\sigma_A} \in B) = \mathbb{P}(T_B \leq T_{A\setminus B}) = \frac{n(B)}{n(A)}$$

by a simple calculation.

3 Brownian excursions and Itô's theorem

Let B a standard BM started from 0.
The local time of B (at 0) is defined by

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t ds \, \mathbb{1}_{|B_s| \leq \varepsilon} \qquad \text{a.s.}$$

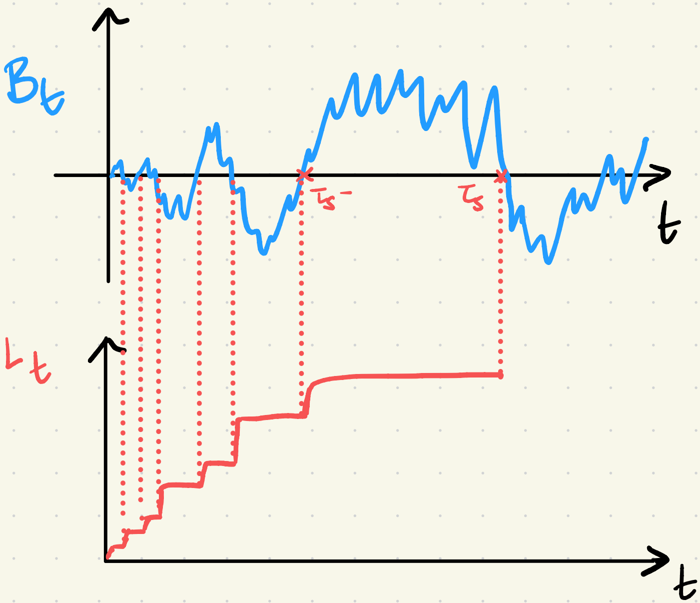
Proposition [Trotter, weak version]
The process $(L_t, t \geq 0)$ is a continuous non-decreasing process. Moreover, the set of increase points of $t \mapsto L_t$ is a.s. equal to

$$\mathcal{Z} = \{t \geq 0, \ B_t = 0\}.$$

Introduce the inverse local time

$$\tau_s := \inf\{t \geq 0, \ L_t > s\}, \qquad s \geq 0.$$

Then $\quad \mathcal{Z} = \{ \tau_{s^-}, \tau_s \ , s \geq 0 \} \quad$ a.s.



Let D the discontinuity set of τ.

The intervals (τ_{s-}, τ_s), $s \in D$, are the connected components of $\{ t \geq 0, B_t \neq 0 \}$. For each of these, one may define an associated excursion :

$$e_s : t \longmapsto B_{\tau_{s^-}+t} \cdot \mathbb{1}_{0 \leq t \leq \tau_s - \tau_{s^-}}$$

Call E the set of excursions, i.e.

$$E := \left\{ e : \mathbb{R}_+ \to \mathbb{R} \text{ continuous s.t. } \begin{array}{l} (i) \ e(0) = 0 \\ (ii) \ \zeta(e) := \inf \{ t > 0, e_t = 0 \} \in (0, \infty) \\ \quad \text{and } e(t) = 0 \quad \forall t \geq \zeta(e) \end{array} \right\}$$

There is a natural distance on E, namely

$$d(e,e') := \sup_{t\geq 0} |e(t)-e'(t)| + |\zeta(e)-\zeta(e')| .$$

Let $\mathcal{E}$ the associated Borel σ-field.

The following theorem, due to Itô, is the start of excursion theory.

Theorem [Itô]

The point measure

$$\sum_{s\in D} \delta_{(s,e_s)}$$

is a Poisson point process on E. Its intensity measure is denoted n and called the Itô excursion measure.

One can write $n = n_+ + n_-$ where n_+ and n_- are carried on the set of positive (E_+) or negative (E_-) excursions respectively.

We stress that n, n_+, n_- are infinite measures due to the contribution of "small" excursions.

④ Properties of Itô's excursion measure

The scaling property of B immediately implies an analogue under n.

Proposition [Scaling property of n]

Let $\lambda > 0$ and define

$$\theta_\lambda : \begin{cases} E \to E \\ e \mapsto \lambda e(t/\lambda^2) \end{cases}$$

Then $\theta_\lambda \circ n = \lambda n$

The following two results are nice applications of PPP techniques.

Proposition [Height of a Brownian excursion]

For $e \in E$, let

$$h(e) = \sup_{s \geq 0} e(s) .$$

For all $x > 0$,

$$n_+(h(e) \geq x) = \frac{1}{2x}$$

Proposition [Duration of a Brownian excursion]

For all $x>0$,
$$n(\zeta(e)\geq x) = \sqrt{\frac{2}{\pi x}}$$

PROOF.

We only prove the statement about the duration (the height is similar).

- First, note that for any $\lambda>0$, and $e\in E$,
$$\zeta(\theta_\lambda(e)) = \lambda^2 \zeta(e).$$

Hence by the scaling property of n,
$$\begin{aligned} n(\zeta(e)\geq x) &= n\left(\tfrac{1}{x}\zeta(e)\geq 1\right) \\ &= n\left(\zeta(\theta_{1/\sqrt{x}}(e))\geq 1\right) \\ &= \frac{1}{\sqrt{x}}\, n(\zeta(e)\geq 1). \end{aligned}$$

- It remains to show that $n(\zeta(e)\geq 1) = \sqrt{\frac{2}{\pi}}$.

Call $C := n(\zeta(e)\geq 1)$.

Note that for all s,
$$\tau_s = \sum_{\substack{r\in D\\ r\leq s}} \zeta(e_r) \qquad \text{a.s.}$$

as a result of the fact that

$\{t \in (0, \tau_s),\ B_t = 0\}$ is Lebesgue-negligible.

Hence by the exponential formula for PPP:

$$\mathbb{E}\left[e^{-\tau_1}\right] = \exp\left(-\int \mathbb{n}(de)\left(1 - e^{-\zeta(e)}\right)\right)$$

It turns out that the Laplace transform of τ_1 is actually known: this gives $\mathbb{E}[e^{-\tau_1}] = e^{-\sqrt{2}}$.

This implies

$$\int \mathbb{n}(de)\left(1 - e^{-\zeta(e)}\right) = \sqrt{2}$$

On the other hand,

$$\begin{aligned}
\int \mathbb{n}(de)\left(1 - e^{-\zeta(e)}\right) &= \int \mathbb{n}(de) \int_0^{\zeta(e)} e^{-u}\, du \\
&= \int_0^\infty du\, e^{-u}\, \mathbb{n}(\zeta(e) \geq u) \\
&= c \int_0^\infty du\, \frac{e^{-u}}{\sqrt{u}} \\
&= \sqrt{\pi}\, c
\end{aligned}$$

Hence $c = \sqrt{2/\pi}$.

Proposition [Markov property under $\mathbb{n}_+$]

The Itô measure $\mathbb{n}_+$ is characterised (among σ-finite measures on E) by the following two properties:

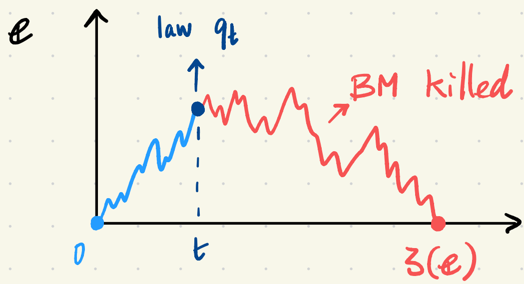
(i) For $t>0$ and $f \in \mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$,

$$n_+\left(f(\mathrm{e}(t))\, \mathbb{1}_{\zeta > t}\right) = \int_0^\infty f(x)\, q_t(x)\, dx$$

where

$$q_t(x) = \frac{x}{\sqrt{2\pi t^3}}\, e^{-x^2/2t}.$$

(ii) Let $t>0$. Under the conditional probability measure $n_+(\cdot \mid \zeta > t)$, the process $(\mathrm{e}(t+r),\ r \geq 0)$ is a BM killed at the origin.



⑤ Normalised Brownian excursions

We now define a variant of the Itô measure n_+, "conditioned on $\zeta = 1$".

Theorem [Disintegration of n_+ over ζ].

There exists a unique collection $(n_{(s)},\ s>0)$ of probability measures on E s.t :

(i) $\forall s > 0 \quad n_{(s)}(\zeta = s) = 1$

(ii) For all $\lambda, s > 0$, $\quad \theta_\lambda \circ n_{(s)} = n_{(\lambda^2 s)}$.

(iii) The disintegration formula holds:

$$n_+ = \int_0^\infty \frac{ds}{2\sqrt{2\pi s^3}} \, n_{(s)} .$$

$n_{(1)}$ is called the law of the normalised Brownian excursion.

NOTE: $\quad$ " $n_{(s)} = n(\cdot \mid \zeta = s)$ "

$\boxed{\text{Theorem}}$ [Finite-dimensional marginals under $n_{(1)}$]

Let $\mathcal{F}_t$ denote the σ-field generated by the coordinate mappings $r \mapsto e(r)$, $r \leq t$. Then for $t \in (0,1)$, $n_{(1)}$ is absolutely continuous w.r.t n_+ on $\mathcal{F}_t$, with Radon-Nikodym derivative

$$\left.\frac{dn_{(1)}}{dn_+}\right|_{\mathcal{F}_t}(e) = 2\sqrt{2\pi} \, q_{1-t}(e(t))$$

In particular, for $0 < t_1 < \dots < t_p < 1$, the law of $(e(t_1), \dots, e(t_p))$ under $n_{(1)}$ has density

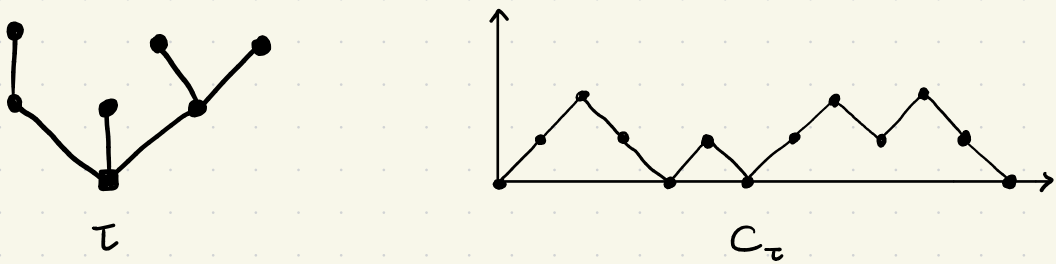
$$2\sqrt{2\pi} \, q_{t_1}(x_1) \, p^*_{t_2-t_1}(x_1, x_2) \cdots p^*_{t_p-t_{p-1}}(x_{p-1}, x_p) \, q_{1-t_p}(x_p)$$

where $p_t^*(x,y)$ are the transition densities of BM killed at 0.

LECTURE 3 - CONVERGENCE OF CONTOUR FUNCTIONS

① Statement

We defined trees τ and their contour functions C_τ:



Theorem For $k \geq 1$, let T_k uniform in $\mathbb{T}_k$, and C_k its contour function. Then

$$\left(\frac{1}{\sqrt{2k}} C_k(2kt), \ 0 \leq t \leq 1 \right) \xrightarrow[k\to\infty]{(d)} \left(\mathbf{e}(t), \ 0 \leq t \leq 1 \right),$$

in $C([0,1], \mathbb{R}_+)$, where $\mathbf{e}$ has law $\mathfrak{n}_{(1)}$.

Recall the definition of the geometric offspring distribution:

$$\mu_0(k) = 2^{-k-1}, \qquad k \in \mathbb{Z}_+ .$$

We saw in [Lecture 1] that we can sample T_k uniform in $\mathbb{T}_k$ from $\mathbb{P}_{\mu_0}(\cdot \mid |t| = k)$.
Hence another way to formulate the above theorem is that contour functions of $BGW(\mu_0)$ trees scale to the normalised Brownian excursion.

② Proof

By [Lecture 1, Section 3], C_k is a SRW excursion conditioned to have length $2k$.
Thus we may rephrase the theorem as follows.
Let $(S_n, n \geq 0)$ a SRW started from $S_0 = 0$.
Define

$$\sigma = \inf\{n \geq 0 : S_n = -1\}.$$

We need to show that under $\mathbb{P}(\quad \mid \sigma = 2k+1)$

$$\left(\frac{1}{\sqrt{2k}} S_{\lfloor \sqrt{2k} t \rfloor}, \ 0 \leq t \leq 1 \right) \xrightarrow{(d)} \left(e(t), \ 0 \leq t \leq 1 \right)$$

normalised BE under $n_{(1)}$

which is a sort of "conditioned Donsker theorem".

To prove this, we prove convergence of marginals and tightness (only sketched here).

CONVERGENCE OF MARGINALS.

Let $i \in \{1, \dots 2k\}$ and $\ell \in \mathbb{Z}_+$.

Then

$$\mathbb{P}(S_i = \ell \mid \sigma = 2k+1) = \frac{\mathbb{P}(S_i = \ell, \sigma = 2k+1)}{\mathbb{P}(\sigma = 2k+1)}.$$

By the simple Markov property at time i,

$$\mathbb{P}(S_i = \ell, \sigma = 2k+1)$$
$$= \mathbb{P}(S_i = \ell, \sigma > i) \cdot \mathbb{P}_\ell(\sigma = 2k+1-i)$$

Now note that $\{S_i = \ell\} \cap \{\sigma > i\}$ is the event that the time-reversed walk $\overleftarrow{S_i}(n) := S_{i-n}$ starting at ℓ does not hit -1 before time i, and satisfies $\overleftarrow{S_i}(i) = 0$.

Since S and $\overleftarrow{S}$ have the same law,

$$\mathbb{P}(S_i = \ell, \sigma > i) = \mathbb{P}_\ell(S_i = 0, \sigma > i)$$
$$= 2\,\mathbb{P}_\ell(\sigma = i+1)$$

$$\left[\mathbb{P}_\ell(\sigma = i+1) = \mathbb{P}_\ell(S_i = 0, \sigma > i, S_{i+1} - S_i = -1) = \tfrac{1}{2}\mathbb{P}_\ell(S_i = 0, \sigma > i)\right]$$

Therefore

$$\mathbb{P}(S_i = \ell \mid \sigma = 2k+1) = \frac{2\, \mathbb{P}_\ell(\sigma = i+1) \cdot \mathbb{P}_\ell(\sigma = 2k+1-i)}{\mathbb{P}(\sigma = 2k+1)}$$

We now use the following classic argument.

$\boxed{\text{Lemma}}$ [Kemperman's formula, weak version]

For all $\ell \in \mathbb{Z}_+$ and $n \geq 1$,

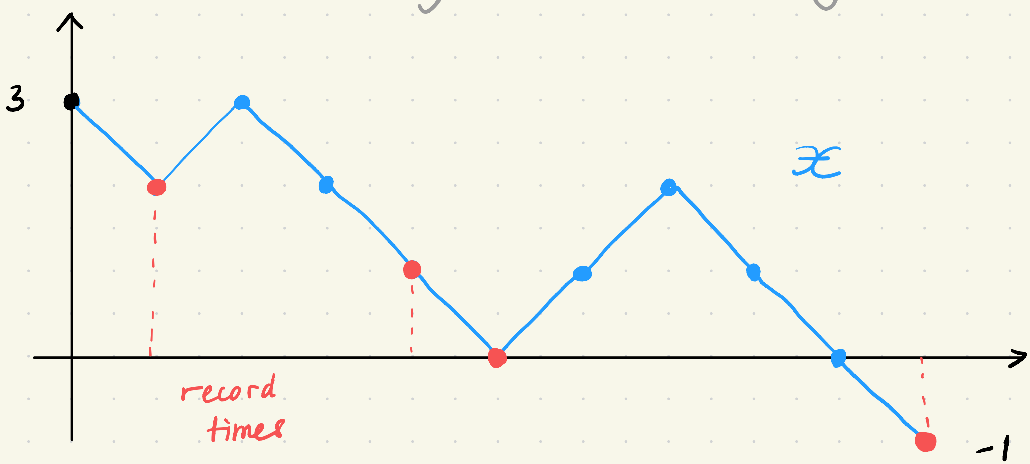
$$\mathbb{P}_\ell(\sigma = n) = \frac{\ell+1}{n} \mathbb{P}_\ell(S_n = -1).$$

PROOF.

Suppose $\mathfrak{X} := (x_1, \dots, x_n)$ is a sequence of increments of the RW so that $S_n = -1$. Then any cyclic shift

$$\mathfrak{X}^{(m)} := (x_m, x_{m+1}, \dots, x_n, x_1, \dots, x_{m-1})$$

also is. How many of them satisfy $\sigma = n$?



We claim that $\sigma = n$ for $\mathcal{X}^{(m)}$ if, and only if, m is a (descending) record time for $\mathcal{X}$, ie the path $\mathcal{X}$ first hits j at time m for some $j \in \{-1, 0, \dots, \ell-1\}$. Indeed, if this is not the case then $\mathcal{X}^{(m)}$ will reach its minimum (which is ≤ -1) before time n.

There are $(\ell+1)$ such record times, hence $(\ell+1)$ shifts $\mathcal{X}^{(m)}$ with $\sigma = n$.

Then since each $\mathcal{X}^{(m)}$ has same probability,

$$\mathbb{P}(\sigma = n \mid S \text{ is a shift of } \mathcal{X}) = \frac{\ell+1}{n}.$$

The result follows by summing over possible $\mathcal{X}$. $\square$

Going back to convergence of marginals, the above lemma implies

$$\mathbb{P}(S_i = \ell \mid \sigma = 2k+1)$$

(*) $$= \frac{2(2k+1)(\ell+1)^2}{(i+1)(2k+1-i)} \cdot \frac{\mathbb{P}_\ell(S_{i+1} = -1)\ \mathbb{P}_\ell(S_{2k+1-i} = -1)}{\mathbb{P}(S_{2k+1} = -1)}$$

We want to prove that

$$\sqrt{2k}\ \mathbb{P}(S_{\lfloor 2kt \rfloor} \in \{\lfloor x\sqrt{2k} \rfloor, \lfloor x\sqrt{2k} \rfloor + 1\} \mid \sigma = 2k+1) \xrightarrow[k \to \infty]{} 4\sqrt{2\pi}\, q_t(x) q_{1-t}(x)$$

uniformly on compact sets of x. Therefore we take $i = \lfloor 2kt \rfloor$ and $\ell \in \{\lfloor x\sqrt{2k} \rfloor, \lfloor x\sqrt{2k} \rfloor + 1\}$ in (*).

We are now left with (unconditional) estimates on S, namely:

$$A_k(x) := \frac{2(2k+1)(\lfloor x\sqrt{2k} \rfloor + 1)^2}{(\lfloor 2kt \rfloor + 1)(2k+1-\lfloor 2kt \rfloor)} \times \frac{1}{\mathbb{P}(S_{2k+1} = -1)},$$

and

$$B_k(x) := \sum_{\ell \in \{\lfloor x\sqrt{2k} \rfloor, \lfloor x\sqrt{2k} \rfloor + 1\}} \mathbb{P}_\ell(S_{\lfloor 2kt \rfloor + 1} = -1)\, \mathbb{P}_\ell(S_{2k+1-\lfloor 2kt \rfloor} = -1).$$

Lemma

Let $p_s(x) := \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s}$.

Then for all $\varepsilon > 0$,

$$\sup_{x \in \mathbb{R}} \sup_{s \geq \varepsilon} \left| \sqrt{n} \cdot \mathbb{P}(S_{\lfloor ns \rfloor} \in \{\lfloor x\sqrt{n} \rfloor, \lfloor x\sqrt{n} + 1 \rfloor\}) - p_s(x) \right| \xrightarrow[n \to \infty]{} 0$$

PROOF.

The law of S_k is given by

$$\mathbb{P}(S_k = y) = \binom{k}{\frac{k+y}{2}} 2^{-k} \qquad y \in \{-k, \ldots, k\} \text{ with same parity as } k.$$

The result follows from explicit calculations. $\square$

Using this Lemma, we see that

$$A_k(x) \underset{k\to\infty}{\sim} 2\sqrt{2\pi}\,\sqrt{\frac{k}{2}}\,\frac{x^2}{t(1-t)}$$

and

$$B_k(x) \underset{k\to\infty}{\sim} \frac{2}{k}\,p_t(x)\,p_{1-t}(x)$$

and thence

$$\sqrt{2k}\,\mathbb{P}\left(S_{\lfloor 2kt\rfloor} \in \{\lfloor x\sqrt{2k}\rfloor, \lfloor x\sqrt{2k}\rfloor+1\} \mid \sigma = 2k+1\right)$$

$$= \sqrt{2k}\cdot A_k(x)\,B_k(x)$$

$$\xrightarrow[k\to\infty]{} 4\sqrt{\pi}\,\frac{x^2}{t(1-t)}\,p_t(x)\,p_{1-t}(x)$$

$$= 4\sqrt{2\pi}\;q_t(x)\,q_{1-t}(x)$$

$$\left[\text{recall } q_t(x) = \frac{x}{t}\,p_t(x)\right]$$

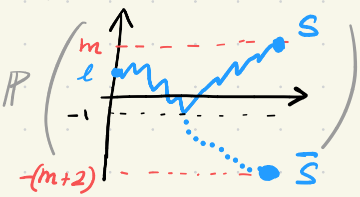
This proves the convergence of first-order marginals. A modification of the argument works for higher order, say 2D marginals for simplicity. For $0<i<j<2k$ and $l, m \in \mathbb{Z}_+$, by the first step in the above argument,

$$\mathbb{P}(S_i = \ell,\ S_j = m,\ \sigma = 2k+1)$$
$$= 2\,\mathbb{P}_\ell(\sigma = i+1)\cdot \mathbb{P}_\ell(S_{j-i} = m,\ \sigma > j-i)\,\mathbb{P}_m(\sigma = k+1-j)$$

The first and last terms are the same as for 1D marginals; it remains to deal with the middle one. But note that

$$\mathbb{P}_\ell(S_{j-i} = m,\ \sigma > j-i)$$
$$= \mathbb{P}_\ell(S_{j-i} = m) - \mathbb{P}_\ell(S_{j-i} = m,\ \sigma \le j-i)$$
$$= \mathbb{P}_\ell(S_{j-i} = m) - \mathbb{P}_\ell(S_{j-i} = -(m+2))$$

reflection principle



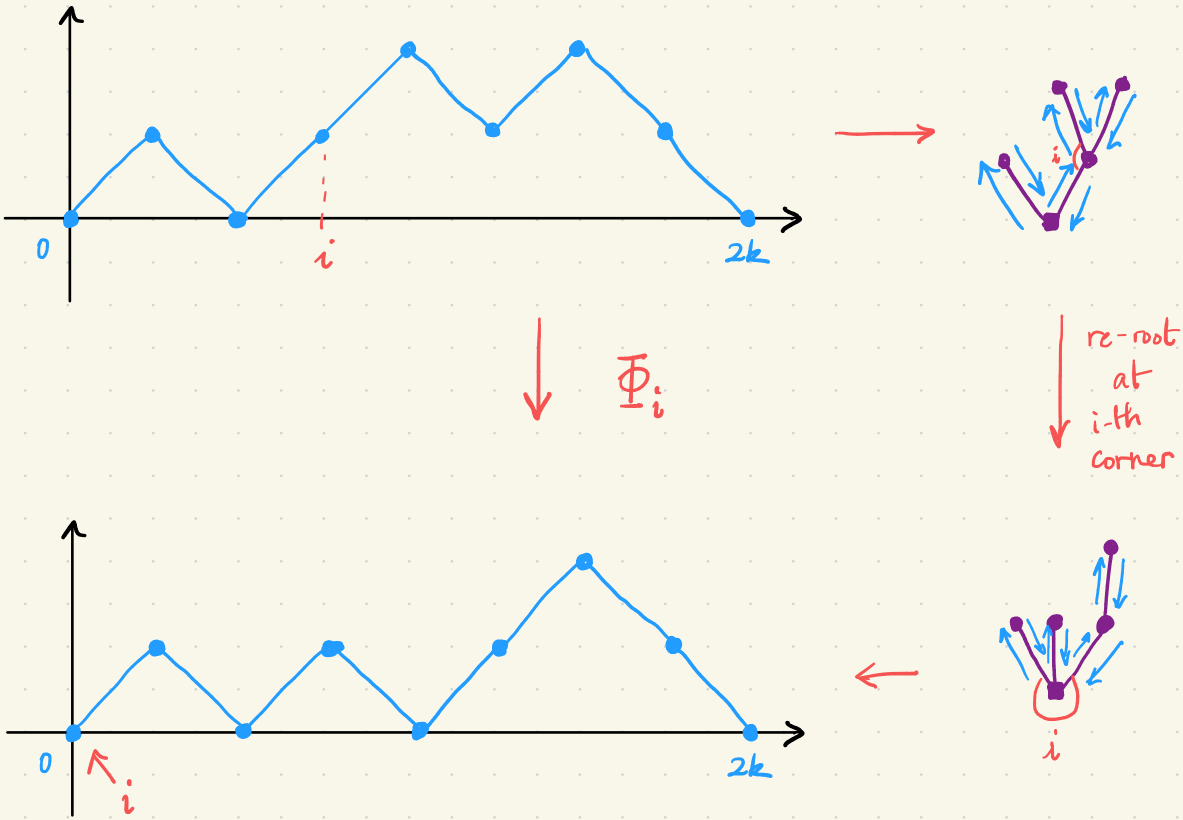
$$= \mathbb{P}(\overline{S}_{j-i} = -(m+2),\ \overline{\sigma} \le j-i)$$

always true when $\overline{S}_{j-i} = -(m+2)$

These quantities are estimated using the Lemma.

TIGHTNESS

Let $(x_0, x_1, \dots, x_{2k})$ a Dyck path of length $2k$, and consider some $i \in \{0, 1, \dots, 2k-1\}$.
One can define another Dyck path $(x_0^{(i)}, \dots, x_{2k}^{(i)})$ "rooted at the i-th corner" as follows:



The mapping Φ_i is a bijection on the set of Dyck paths with length $2k$.

A formal definition of Φ_i is:

$$x_j^{(i)} := x_i + x_{i\oplus j} - 2 \min_{i\wedge(i\oplus j)\le n\le i\vee(i\oplus j)} x_n .$$

where $i\oplus j = \begin{cases} i+j & \text{if } i+j\le 2k \\ i+j-2k & \text{if } i+j>2k \end{cases}$.

Denote
$$\underline{C}_k^{i,j} = \min_{i\wedge j\le n\le i\vee j} C_k(n).$$

By the previous discussion, for any $i\in\{0,\dots,2k\}$,

$$\left(C_k(i) + C_k(i\oplus j) - 2\,\underline{C}_k^{i,\,i\oplus j},\quad 0\le j\le 2k\right)$$
$$\stackrel{(d)}{=}$$
$$\left(C_k(j),\quad 0\le j\le 2k\right) \qquad (*)$$

Coming back to the proof of tightness, we want to check Kolmogorov's tightness criterion.

For $0\le i<j\le 2k$, and $p\in\mathbb{Z}_+$,

$$\mathbb{E}\left[\left(C_k(j) - C_k(i)\right)^{2p}\right]$$
$$\le \mathbb{E}\left[\left(C_k(j) + C_k(i) - 2\,\underline{C}_k^{i,j}\right)^{2p}\right]$$

By (*) above, we get

$$\mathbb{E}[(C_k(j)-C_k(i))^{2p}] \leqslant \mathbb{E}[C_k(j-i)^{2p}]$$

Assume:

FACT: $\mathbb{E}[C_k(i)^{2p}] \leqslant K_p \cdot i^p$

Then we have obtained

$$\mathbb{E}[(C_k(j)-C_k(i))^{2p}] \leqslant K_p (j-i)^p .$$

Therefore

$$\mathbb{E}\left[\left(\frac{C_k(2kt)-C_k(2ks)}{\sqrt{2k}}\right)^{2p}\right] \leqslant K_p (t-s)^p$$

for all $s,t \in [0,1]$.

[actually only for $s=\frac{i}{2k}$, $t=\frac{j}{2k}$, but we can extend this since C_k is 1-Lipschitz]

This concludes the proof of tightness (modulo the technical lemma).

LECTURE 4 - CONVERGENCE OF REAL TREES

① Real trees

Definition

A *real tree* is a compact metric space (T, d) such that for all $a, b \in T$:

(i) There exists a *unique isometric mapping* $f_{a,b} : [0, d(a,b)] \to T$ s.t. $f_{a,b}(0) = a$

$f_{a,b}(d(a,b)) = b$

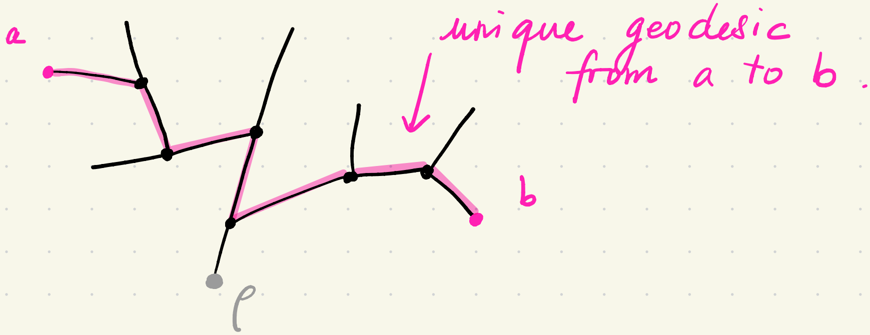
(ii) For any continuous injection $q : [0,1] \to T$ with $q(0) = a$ and $q(1) = b$, we have

$$q([0,1]) = f_{a,b}([0, d(a,b)])$$

In the sequel, we shall only consider *rooted* real trees, i.e. we have a distinguished vertex $\rho = \rho(T)$ called the root.

INTERPRETATION.

One should think of $(\mathcal{T}, d)$ as a union of line segments with no cycles.



$d(a,b) =$ "length" of pink path.

VIEWPOINT: "How to find genealogies in $(\mathcal{T}, d)$?"

Denote by

$$[\![a,b]\!] = f_{a,b}([0, d(a,b)])$$

the trace of the path between a and b.

We interpret $[\![\rho, a]\!]$ as the ancestral lineage of a. Any $b \in [\![\rho, a]\!]$ is called an ancestor of a, and we write $b \preceq a$.

[Note that $\preceq$ is a partial order on $\mathcal{T}$.]

Moreover, for all $a, b \in \mathcal{T}$, there exists a unique $c := a \wedge b \in \mathcal{T}$ s.t.

$$[\![\rho, a]\!] \cap [\![\rho, b]\!] = [\![\rho, c]\!]$$

[Exercise : prove it !]

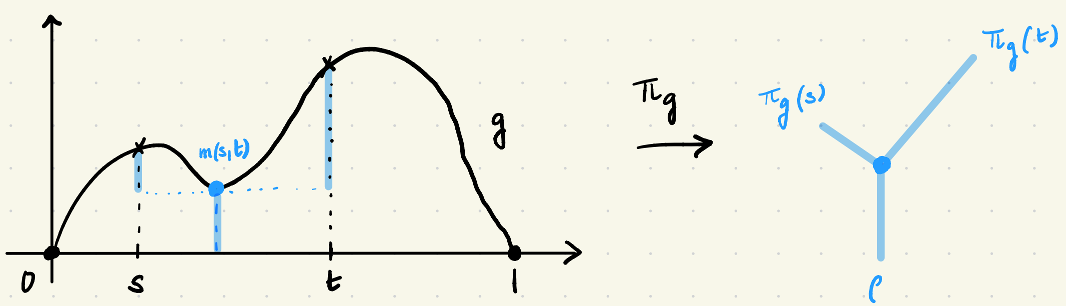
$a \wedge b$ is the most recent common ancestor of a, b.

We sometimes call vertices the elements of $\mathcal{T}$. The multiplicity of $a \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \setminus \{a\}$. Leaves are vertices with multiplicity 1.

② Contour of real trees

We now describe a way to obtain real trees from excursion functions.

Let $g : [0,1] \to \mathbb{R}_+$ $(g \neq 0)$ with $g(0) = g(1) = 0$.



For $s, t \in [0,1]$, we set

$$m(s,t) := \inf_{r \in [s \wedge t, s \vee t]} g(r)$$

Let

$$d_g(s,t) = g(s) + g(t) - 2\, m(s,t)$$

d_g is a pseudometric on $[0,1]$.

Introduce the equivalence relation

$$s \sim t \iff d_g(s,t) = 0$$

$$(\iff g(s) = g(t) = m(s,t))$$

Then we define

$$T_g = [0,1]/\sim$$

Let

$$\pi_g : [0,1] \longrightarrow T_g$$

the canonical projection.

$\boxed{\text{Theorem}}$ (T_g, d_g) is a real tree.

We may view it as a rooted tree with root $\rho = \pi_g(0) = \pi_g(1)$.

③ The Gromov-Hausdorff topology

We need to make sense of convergence of compact metric spaces.

Let (E, δ) a metric space. For two compacts $K, K' \subset E$, there is a notion of distance, namely

$$\delta_{Haus}(K, K') := \inf\{\varepsilon > 0,\ K \subset B_\varepsilon(K') \text{ and } K' \subset B_\varepsilon(K)\},$$

where $B_\varepsilon(X) := \{y \in E,\ \delta(y, X) \leq \varepsilon\}$

Now if (E_1, ρ_1) and (E_2, ρ_2) are two pointed compact metric spaces, we define the Gromov-Hausdorff distance:

$$d_{GH}(E_1, E_2) := \inf_{\varphi}\{\delta_{Haus}(\varphi_1(E_1), \varphi_2(E_2)) \vee \delta(\varphi_1(\rho_1), \varphi_2(\rho_2))\}$$

Here $\inf\limits_{\varphi}$ is over all metric spaces (E, δ) and all isometric embeddings $\varphi_1 : E_1 \to E$

$$\varphi_2 : E_2 \to E$$

We say that E_1 and E_2 are equivalent if there is an isometry between them sending ρ_1 to ρ_2.
Noting that $d_{GH}(E_1, E_2)$ only depends on the equivalence classes of E_1 and E_2, we introduce

$\mathbb{K}$ = space of equivalent classes of pointed compact metric spaces.

Theorem

d_{GH} is a metric on $\mathbb{K}$ and the space $(\mathbb{K}, d_{GH})$ is separable and complete.

The following crucial bound reduces the convergence of trees to that of contour functions.

Theorem

Let g, g' two excursion functions

Then

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g-g'\|_\infty.$$

In particular $g \mapsto \mathcal{T}_g$ is continuous.

④ The continuum random tree (CRT)

Let $\mathbb{e}$ a normalised Brownian excursion under $n_{(1)}$.

Definition

The Brownian continuum random tree (CRT) is the random real tree $\mathcal{T}_{\mathbb{e}}$. It is a random variable in $(\mathbb{K}, d_{GH})$.

Theorem

For $k \geq 1$, let T_k uniform in $\mathbb{T}_k$. We see T_k as a metric space with the graph distance d_k on T_k. Then, we have the convergence in distribution:

$$\left(T_k, \frac{1}{\sqrt{2k}} d_k\right) \xrightarrow[k \to \infty]{d} \left(\mathcal{T}_{\mathbb{e}}, d_{\mathbb{e}}\right)$$

in the metric space $(\mathbb{K}, d_{GH})$.

PROOF.

Recall that C_k denotes the contour function of T_k, extended as a function on $[0,1]$.

Now define

$$\tilde{C}_k : t \in (0,1) \longmapsto \frac{1}{\sqrt{2k}} \, c_k(2kt)$$

Notice that $\tilde{C}_k$ is an excursion function, and as such, defines a real tree $(\tilde{T}_k, \tilde{d}_k)$

- On the one hand, we proved in [Lecture 3] that

$$\tilde{C}_k \xrightarrow{(d)} e$$

in the uniform topology. The bound

$$d_{GH}(T_g, T_{g'}) \le 2\|g-g'\|_\infty,$$

entails by the continuous mapping theorem that

$$(\tilde{T}_k, \tilde{d}_k) \xrightarrow{(d)} (T_e, d_e)$$

in the space $(\mathbb{K}, d_{GH})$.

- On the other hand, $\tilde{T}_k$ is isometric to a finite union of line segments of length $\frac{1}{\sqrt{2k}}$, which are glued according to the genealogies of T_k. Therefore, by definition of d_{GH},

$$d_{GH}\left(\left(T_k, \frac{1}{\sqrt{2k}} d_k\right), \left(\tilde{T}_k, \tilde{d}_k\right)\right) \leq \frac{1}{\sqrt{2k}}.$$

These two facts prove our claim. $\square$

LECTURE 5 - A SENSE OF PLANAR MAPS

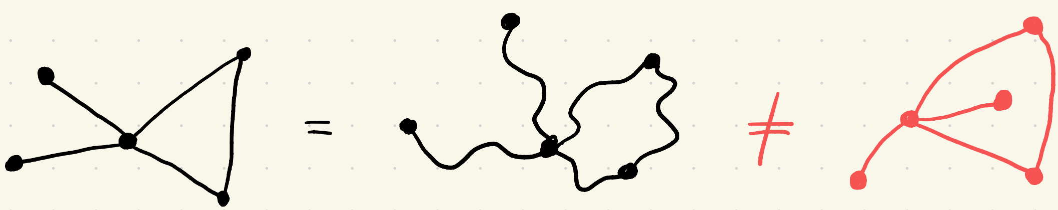
① Planar maps

Definition

A planar map is a finite connected graph drawn on the sphere $\mathbb{S}^2$ without edge crossings. We view them up to orientation-preserving homeomorphisms.

We allow graphs to have multiple edges or loops.

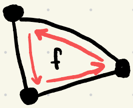
EXAMPLE.

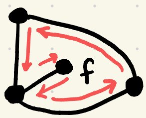


[Note that these are the same as graphs though]

For symmetry reasons, it is convenient to consider rooted planar maps, i.e. we will have a

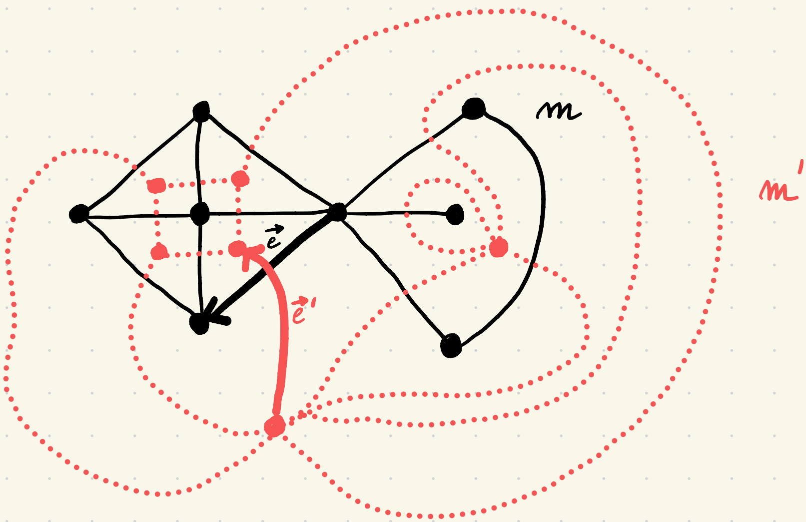
distinguished oriented edge of the map denoted $\vec{e}$.
In the sequel, all planar maps will be rooted.
The degree $\deg(f)$ of a face f is the number of edges counted when drawing the (inner) contour of f:

deg () = 3

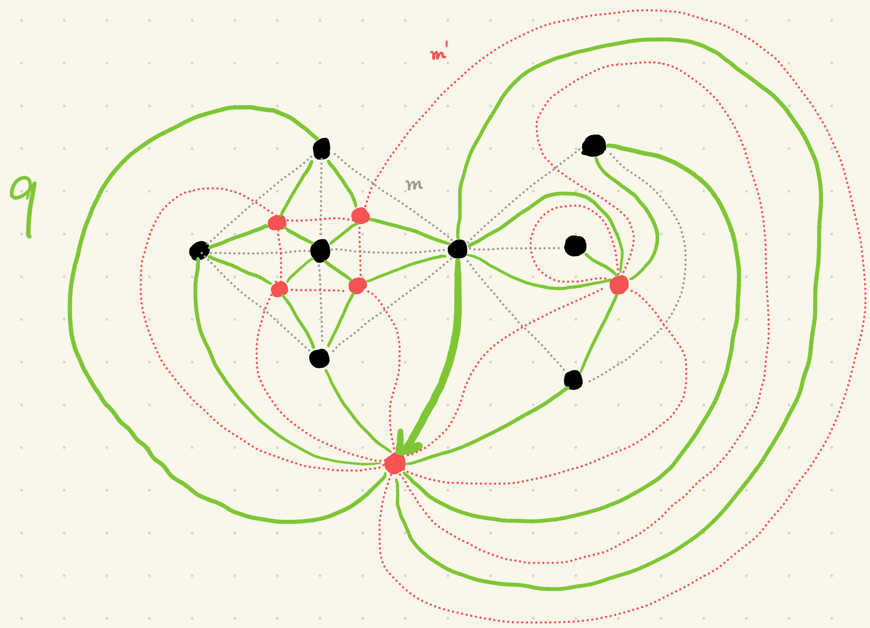
deg () = 5

DUALITY AND THE TUTTE BIJECTION.

Given a planar map m, we can construct the dual map m' as follows:



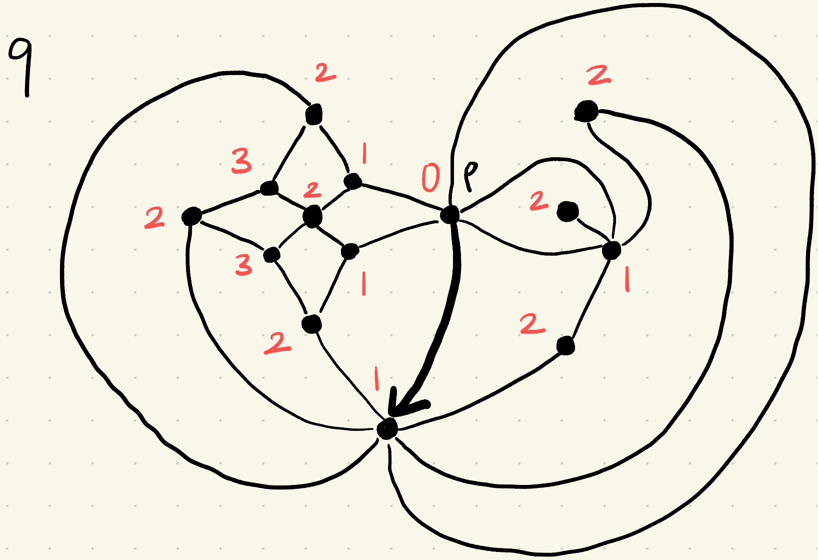
The Tutte bijection is a bijection between planar maps with n edges and quadrangulations with n faces. A quadrangulation is a planar map whose faces are all of degree 4. Here is how the bijection works (compare with previous drawing):



The quadrangulation q is in green.

② The Cori-Vauquelin-Schaeffer bijection

Consider a rooted quadrangulation q:



We will see how to construct a (labelled) tree out of q.

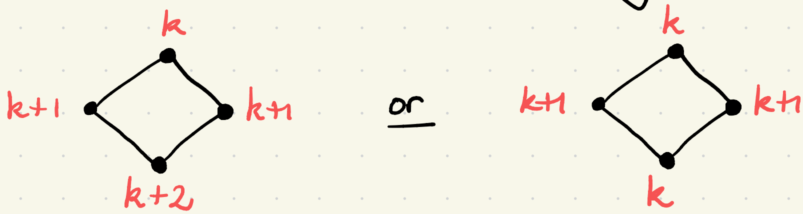
Let ρ be the root vertex of the map (ie. the origin of the vector $\vec{e}$).

Label each vertex of the map by the distance to ρ (see picture above): we denote by $\phi(v)$ the label of vertex v.

Notice that if v and w are connected by an edge, $\phi(v) - \phi(w) \in \{-1, 1\}$

Furthermore, observe the :

FACT - Faces are of the following form :

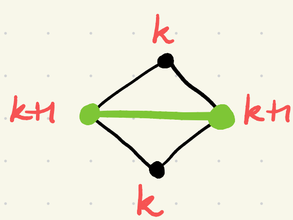


"simple" "confluent"

(i.e. at least one pair of opposite edges have the same label)

We construct a subset of green edges on top of q by looking around each face as follows :

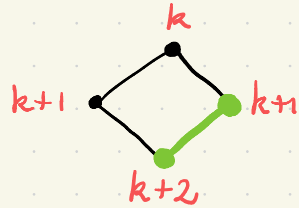
- for confluent faces:



ie.

"join the vertices with maximal labels."

- for simple faces :

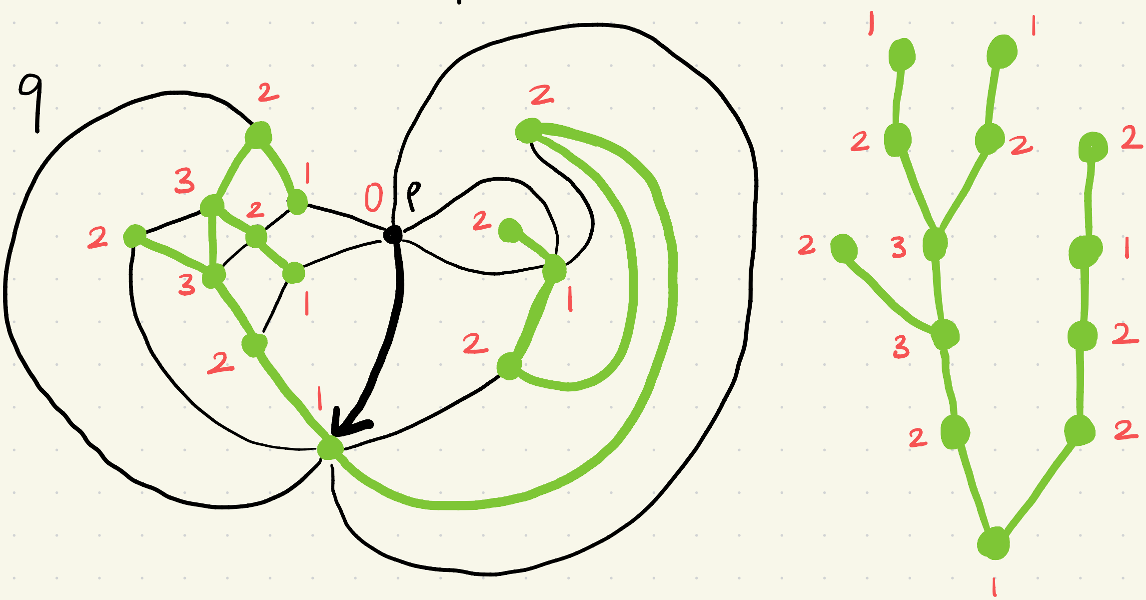


ie.

"look at the vertex v with maximal label, and choose the edge (v, w) of q leaving the face to the left"

EXAMPLE.

Let's go back to the original map and run the above procedure!



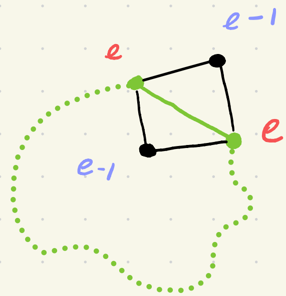
We call $T(q)$ the resulting graph. We may root it by declaring the target of $\vec{e}$ to be the root of $T(q)$.

Theorem [Cori-Vauquelin, Chassaing-Schaeffer]
The resulting graph $T(q)$ is a (rooted) tree. Moreover, it defines a bijection between rooted quadrangulations with n faces and well-labelled trees with n edges.
[A tree is said to be well-labelled if all its vertices have labels in $\mathbb{N}^*$, with root label 1, and the labels of two neighbouring vertices differ by at most 1.]

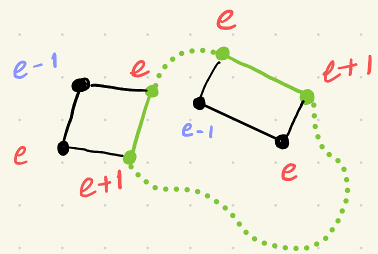
In probabilistic terms, it is very easy to sample a uniform quadrangulation with n faces: just sample a well-labelled tree with n edges uniformly at random.

Why is $T(q)$ a tree?

Suppose there exists a cycle in $T(q)$. Let e the minimal label around that cycle. Then either all the labels around the cycle are e, or we can find two edges with labels $(e, e+1)$ and $(e+1, e)$, i.e



CASE 1



CASE 2

In any case, note that we can find a vertex with label $e-1$ both "outside" and "inside" the cycle. This is impossible since labels are distances from the root, and the minimal label around the cycle is e.

REMARK.

- The bijection works actually better with pointed maps, where we have another distinguished vertex.
- A corollary of the CVS bijection is the following enumeration result:

Let M_n be the set of planar maps with n edges
Q_n ——— quadrangulations with n faces
[Recall that all planar maps are rooted here]
Then

$$\# M_n \underset{\substack{\uparrow \\ \text{Tutte}}}{=} \# Q_n = \frac{2}{n+2} 3^n \, Cat_n .$$

- Remarkably, it keeps track of metric properties of the quadrangulation: labels on the tree record distances from a distinguished vertex.
- Finally, there is an extension of the previous bijection to bipartite planar maps, due to

Bouttier, Di Francesco and Guitter. This is in particular relevant for other models of random planar maps coupled to a statistical physics model.

THE END