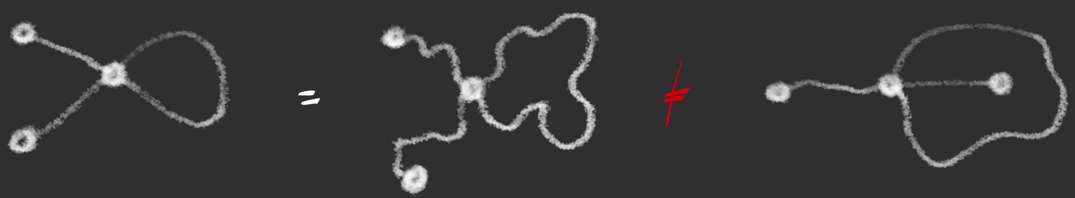
THE SCALING LIMIT OF THE VOLUME OF LOOP-O(2) QUADRANGULATIONS

1. General overview on rigid loop-O(n) quadrangulations

- PLANAR MAPS :

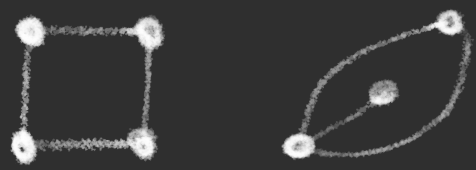


rooted

$$\text{perimeter} = \deg(F_{ext})$$

- QUADRANGULATIONS :

  All internal faces have degree 4



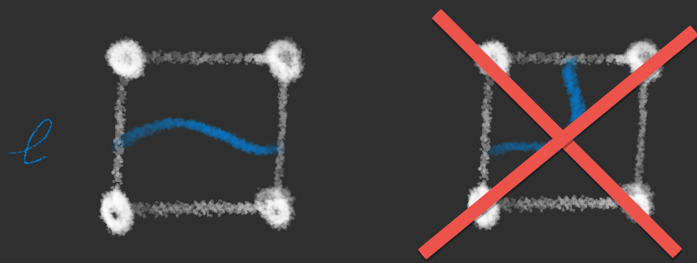
QUESTION

How does the volume scale with perimeter?

- LOOPS :

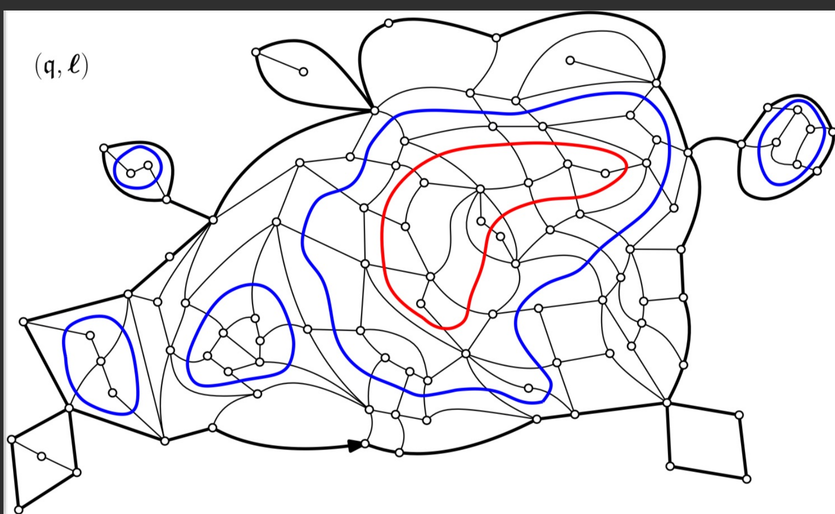
  living on the faces of quadrangulation, disjoint and nested.

RIGIDITY condition :



Notation

$$\mathcal{O}_p := \{ \text{rigid loop-decorated quadrangulations with perimeter } 2p \}$$



- LOOP-O(n) MODEL : $n \in (0,2],\ g, h \geq 0$

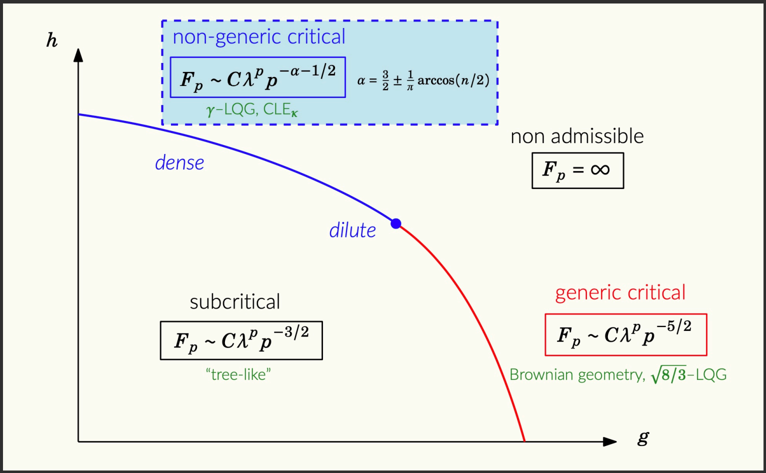
weight $$w(\mathfrak{q}, \ell) := g^{\#\square}\, h^{\#\square}\, n^{\#\text{loops}}$$

partition function $$F_p := \sum_{(\mathfrak{q},\ell)\in \mathcal{O}_p} w(\mathfrak{q},\ell)$$

probability measure $$\mathbb{P}^{(p)}(\mathfrak{q},\ell) := \frac{w(\mathfrak{q},\ell)}{F_p}$$

when $F_p < \infty$

- PHASE DIAGRAM.



$\boxed{n < 2}$

- Main tool = "gasket decomposition"
- Can take limit $n \to 2^-$ to get a "non-generic critical" model.

$$F_p \sim \frac{c\lambda^p}{p^2} \quad \text{OR} \quad F_p \sim \frac{c\lambda^p}{p^2}\log(p)$$

2. The volume of loop-O(n) quadrangulations: main results

$(n; g, h) \in \mathcal{D} \leftarrow$ non-generic critical line

VOLUME = number of vertices.

MEAN ASYMPTOTICS [Budd]

$$\bar{V}(p) = \mathbb{E}^{(p)}_{(n;g,h)}[V] \sim \Lambda p^{\theta_\alpha}$$

where

$$\theta_\alpha = \min(2, 2\alpha - 1)$$

Remark: $\theta_\alpha = \begin{cases} 2 & \text{dilute} \\ 2\alpha - 1 & \text{dense} \end{cases}$

For $n=2$ we establish

$$\bar{V}(p) \sim \Lambda p^2 \quad \text{OR} \quad \bar{V}(p) \sim \Lambda \frac{p^2}{\log(p)}.$$

MAIN RESULT

THEOREM [Aïdékon, DS, Hu '24]

(i) When $n \in (0,2)$,

$$\frac{V}{\bar{V}(p)} \xrightarrow{(d)} W_\infty$$

(ii) When $n = 2$,

$$\log(p) \cdot \frac{V}{\bar{V}(p)} \xrightarrow{(d)} D_\infty$$

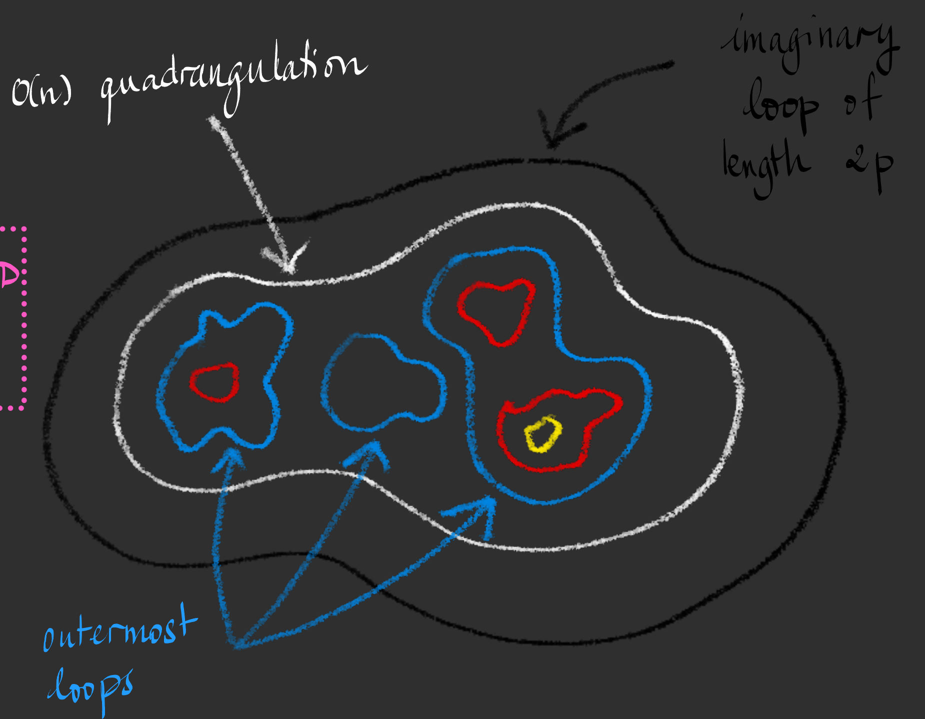
CONJECTURED BY

(i) Chen - Curien - Maillard

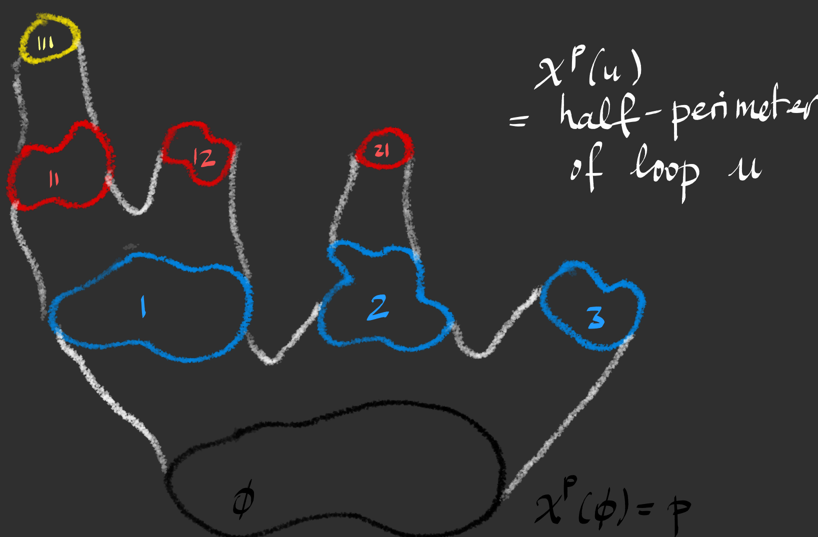
(ii) Aïdékon - DS

3. Some proof ideas

LOOP-DECORATED MAP



CASCADE OF NESTED LOOPS



THM [Chen-Curien-Maillard '17]

For $n \neq 2$,

$$\frac{1}{p}\left(\chi^p(u),\, u \in \mathcal{U}\right) \xrightarrow{d} \left(Z_\alpha(u),\, u \in \mathcal{U}\right)$$

[α is the same as in the asymptotics of F_p]

DESCRIPTION OF Z_α :

$$Z_\alpha(u) = \exp\left(BRW_\alpha(u)\right)$$

In the discrete :

Around any vertex, we see a Markov chain corresponding to the (half) perimeters of the loops around that vertex.

In the continuum :

We see a random walk around any point.

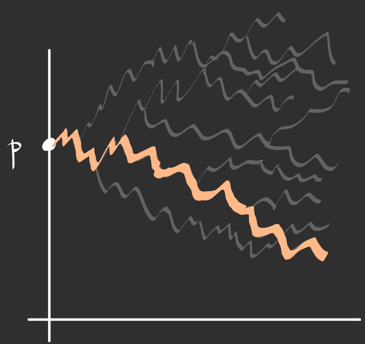
How do they behave ?

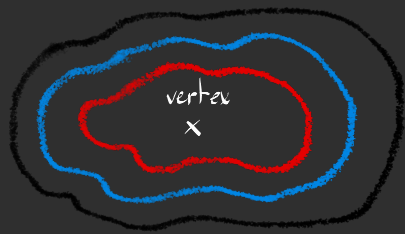
HEURISTICS FOR OUR MAIN RESULT

$$p^{-\theta_\alpha} V(p) \approx p^{-\theta_\alpha} \sum_{|u|=\ell} V(u)$$

"volume outside loops at generation ℓ doesn't count"

$$\uparrow$$





sudden drop to 0 before generation ℓ

$$\approx p^{-\theta_\alpha} \sum_{|u|=\ell} \overline{V}(\chi^\ell(u))$$

"concentration"

$$\approx \Lambda \sum_{|u|=\ell} \left(\frac{\chi^\ell(u)}{p}\right)^{\theta_\alpha}$$

"asymptotics of $\overline{V}(q)$"

$$\approx \Lambda \sum_{|u|=\ell} Z_\alpha(u)^{\theta_\alpha}$$

"convergence of the cascade"

$$\downarrow \ell \to \infty$$

$$\Lambda W_\infty$$

COMMENTS

(a) Concentration is hard!

Second moment blows up:

$$\mathbb{E}[\mathfrak{Z}^2] = \infty$$

duration of
Brownian cone excursion.

$\Longrightarrow$ CLASSIFICATION INTO

$$V = V_{good} + V_{bad} .$$

(b) When $n = 2$, $W_\infty = 0$ a.s, so
we end up with

$$p^{-2} V(p) \xrightarrow[p \to \infty]{} 0 .$$

Need to introduce the

DERIVATIVE MARTINGALE.

FEATURES OF THE CONTINUUM CASCADE Z_α :

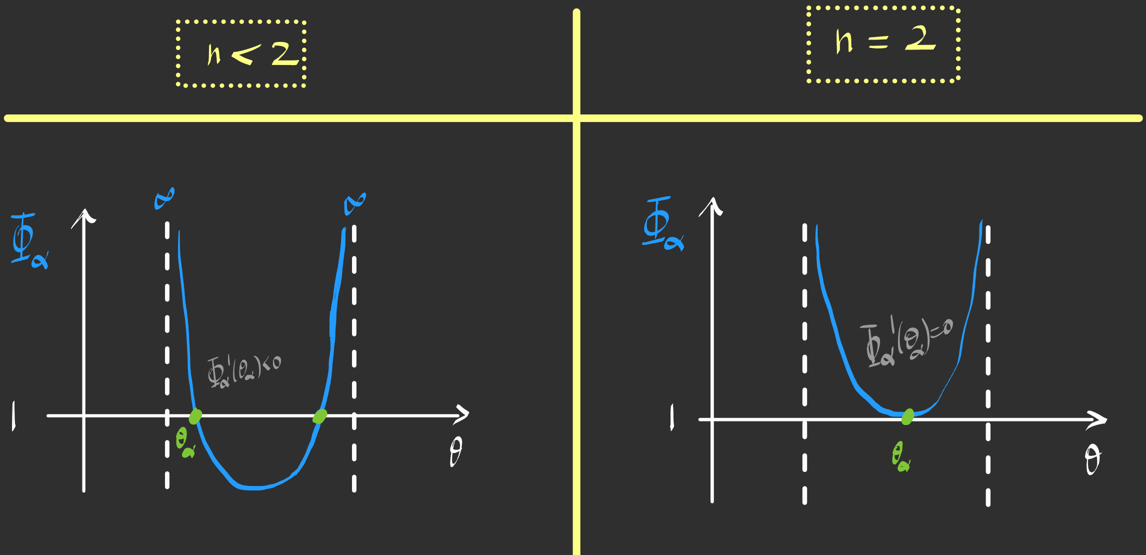
$$\Phi_\alpha(\theta) := \mathbb{E}\Big[\sum_{|u|=1} Z_\alpha(u)^\theta\Big]$$

$\approx$ Laplace transform of underlying BRW

When $\Phi_\alpha(\theta) = 1$,

$$M_k := \sum_{|u|=k} Z_\alpha(u)^\theta, \qquad k \geq 0,$$

is a martingale.



Define θ_α = minimal root.

Introduce tagged particle:

$$Z_n^* = \exp\left(\sum_{i=1}^{k} \xi_i^*\right) \qquad \text{iid } \xi_i^*$$

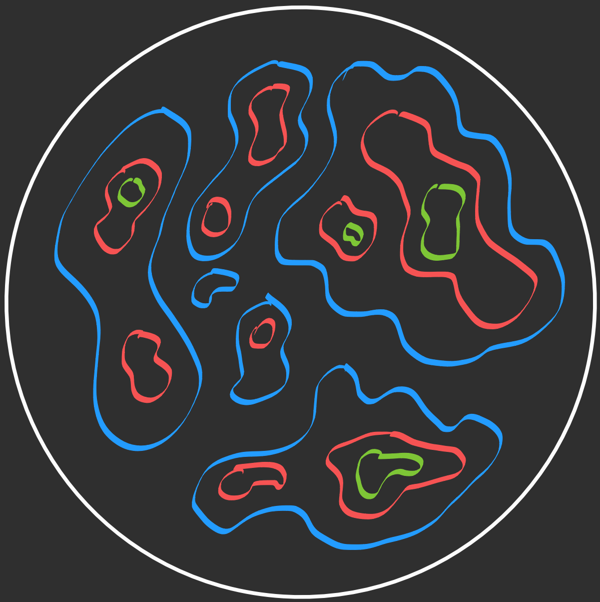
$$\mathbb{E}[F(\xi^*)] = \mathbb{E}\Big[\sum_{i=1}^{\infty} \underbrace{Z_\alpha(i)^{\theta_\alpha}}\, F(\log Z_\alpha(i))\Big]$$

w.p. $Z_\alpha(i)^{\theta_\alpha}$,

Z^* follows particle i

INTERPRETATION:

Z^* describes behavior of a typical particle.



At each nesting level, pick one of the loops proportional to its (conditional) quantum area $\approx Z_\alpha(i)^{\theta_\alpha}$

How does Z^* behave?

Note: $\mathbb{E}[\xi^*] = \Phi_\alpha'(\theta_\alpha)$

- $n \in (0,2)$: $\mathbb{E}[\xi^*] < 0$

$\Rightarrow$ typically, particles decay exponentially.

- $n = 2$: $\mathbb{E}[\xi^*] = 0$

$\Rightarrow$ not clear what happens.

For $n=2$ the BRW is critical.

In that case:

- use a truncation argument $(\chi^P(u) < B)$ to get size decay.

- remove truncation

logarithmic cost for Y^* to stay below a barrier:

$$\mathbb{P}(T_a < T_b^+) \approx \frac{\ln b}{\ln a} \qquad \begin{array}{l} a \to 0 \\ b \to \infty \end{array}$$

BUT

Need to deal with estimates for the discrete cascade (not the BRW)!

Use a coupling argument.