The LIS of Brownian separable permutons

joint work with Adhikari
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1. THE BROWNIAN SEPARABLE PERMUTONS

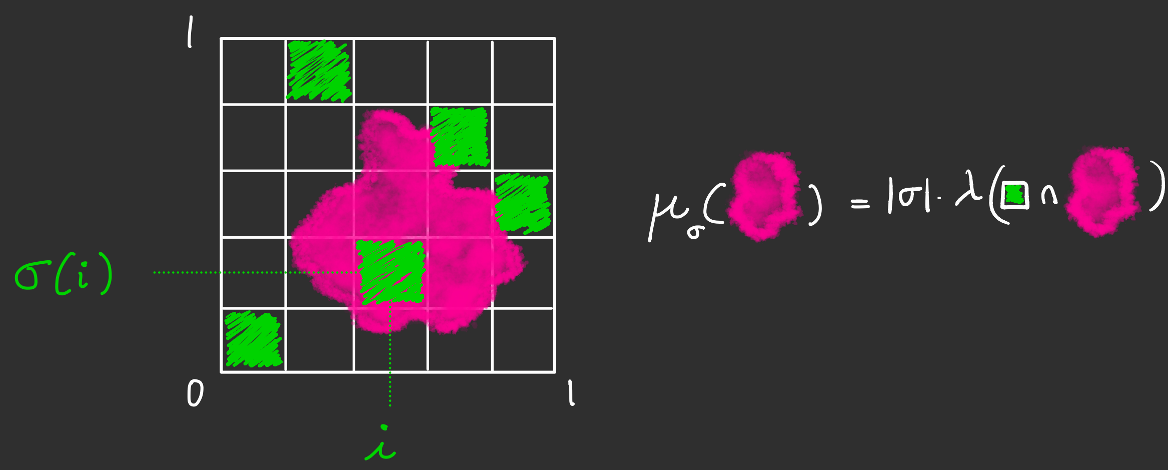
« Universal scaling limit of pattern-avoiding permutations »

What is the "limit" of a permutation ?

DEF. A permuton μ is a probability measure on $[0,1]^2$ with uniform marginals:

$$\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a$$

Given a permutation σ, we construct a permuton μ_σ :



Let (σ_n) a sequence of permutations with $|\sigma_n| = n$.

We say that $(\sigma_n)_{n\geq 1}$ converges to a permuton μ if

$$\mu_{\sigma_n} \longrightarrow \mu \qquad \text{weakly}$$

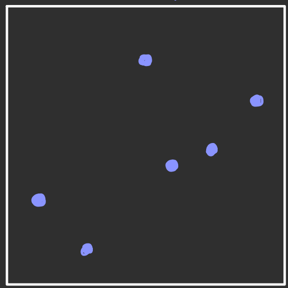
"DEF." [Bassino et al., Maazoun, ...]
The Brownian separable permutons are a family $(\mu_p)_{p\in(0,1)}$ of permutons arising as the limit of many (pattern-avoiding) permutations

First RANDOM limiting permutons

Description of $(\mu_p)_{p\in(0,1)}$ -

Given a permuton μ, we can sample a permutation $\mathrm{Perm}[\mu, n]$ of size n by drawing n iid points (Z_i) according to μ:

$Z_i \sim \mu$

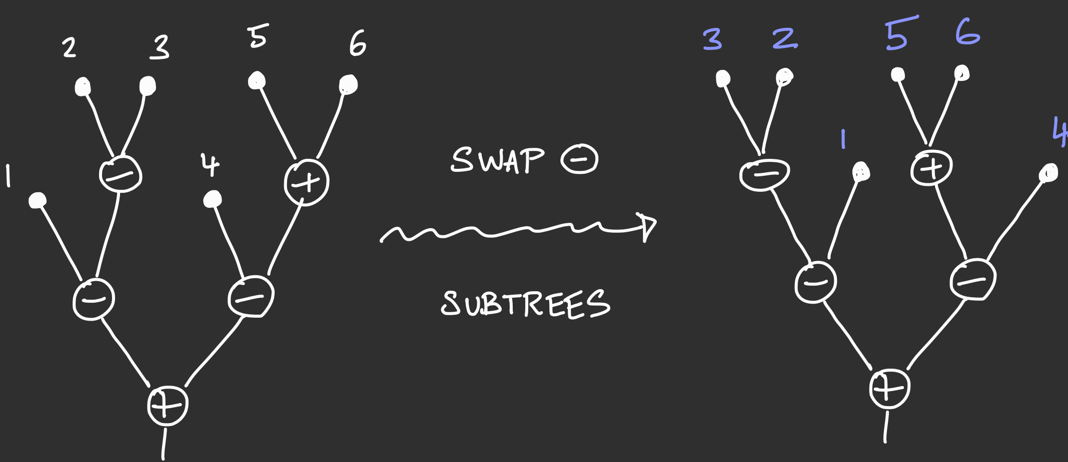


induces a permutation

$\mathrm{Perm}[\mu, n] = 217345$.

We will only be interested in $\mathrm{Perm}[\mu_p, n]$, so let's describe that (instead of μ_p).

$T_n(p)$ uniform binary tree with n leaves and i.i.d. $\mathrm{Ber}(p)$ ⊕/⊖ spins on its nodes

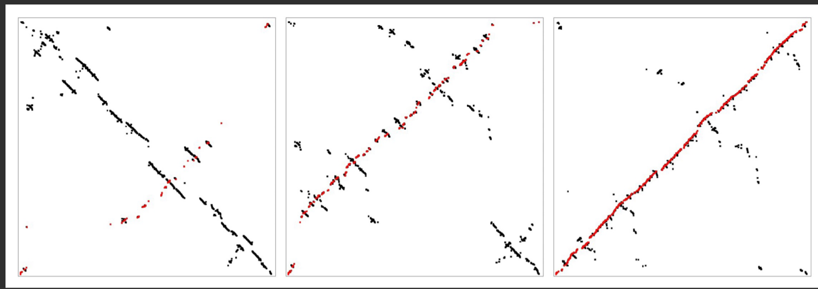


$T_n(p)$

$\sigma = 321564$

$\sim \mathrm{Perm}[\mu_p, n]$

Simulations of $\mathrm{Perm}[\mu_p, n]$



$p = 0{,}2$ $\quad$ $p = 0{,}5$ $\quad$ $p = 0{,}9$

2. THE ULAM-HAMMERSLEY PROBLEM

Let $\mathrm{LIS}(\sigma)$ the size of the longest increasing subsequence of a permutation σ.

$$\sigma = 321564 \rightsquigarrow \mathrm{LIS}(\sigma) = 3$$

What is the behaviour of $\mathrm{LIS}(\sigma_n)$ as $n \to \infty$?

Remark: [Romik's book]

For uniform permutations $\sigma_n \in \mathfrak{S}_n$,

$$\mathrm{LIS}(\sigma_n) \approx \sqrt{n} \quad \text{as } n \to \infty.$$

Typically $\mathrm{LIS}(\sigma_n) \simeq \sqrt{n}$ or $\mathrm{LIS}(\sigma_n) \approx n$.

What about pattern-avoiding permutations?

THM [Borga, DS, Gwynne '23]

Let $p \in (0,1)$ and $\sigma_n = \mathrm{Perm}[\mu_p, n]$.
There exist explicit $\alpha_*(p), \beta^*(p) \in (\frac{1}{2}, 1)$
s.t.

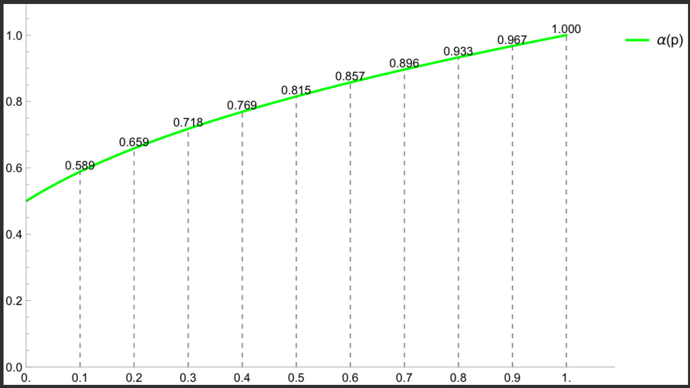
$$n^{\alpha_*(p) - o(1)} \leq \mathrm{LIS}(\sigma_n) \leq n^{\beta^*(p) + o(1)}$$

with high probability.

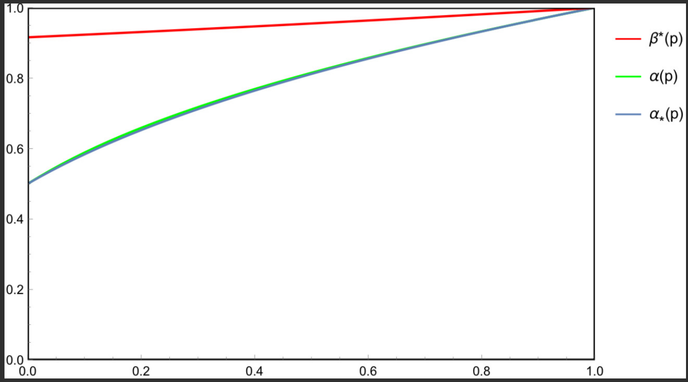
3. MAIN RESULT

SIMULATIONS.

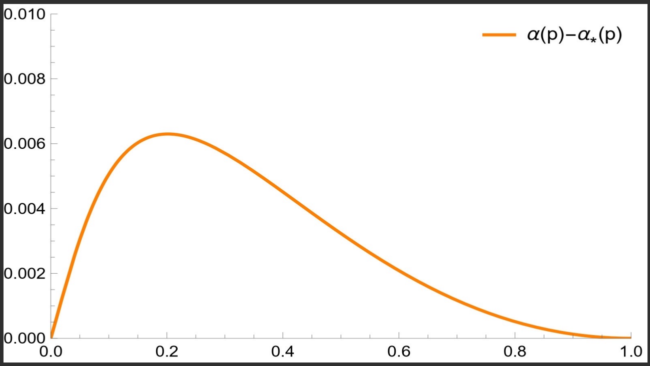
Simulation of "$\log(\mathrm{LIS}(\sigma_n))$" as a function of $p \in (0,1)$



Simulation of "$\log(\mathrm{LIS}(\sigma_n))$" and the exponents $\alpha_*(p)$ and $\beta^*(p)$



Simulation of the gap



THM [Adhikari, Borga, Budzinski, DS, Sénizergues '25]
Let $p \in (0,1)$ and $\sigma_n = \mathrm{Perm}[\mu_p, n]$.
There exists an a.s. positive and finite random vaiable X s.t.:

$$\frac{\mathrm{LIS}(\sigma_n)}{n^{\alpha(p)}} \xrightarrow{a.s.} X$$

where $\alpha(p) \in (\frac{1}{2}, 1)$ solves

$$\frac{1}{4^{1/2\alpha}\sqrt{\pi}} \cdot \frac{\Gamma(\frac{1}{2} - \frac{1}{2\alpha})}{\Gamma(1 - \frac{1}{2\alpha})} = \frac{p}{p-1}$$

COMMENTS

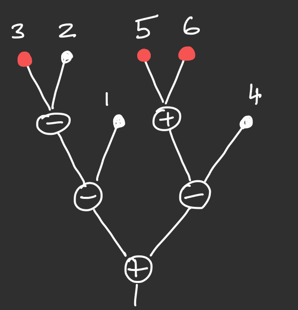
- This is much stronger than a statement "whp": we have a scaling limit result.
- The same result holds for $\mathrm{LPS}(T_n)$, $\mathrm{LCL}(G_n)$, $\mathrm{LDP}(M_n)$
- $X = X(p)$ is a deterministic function of μ_p

You should care because:

1) There are very beautiful math already for uniform models (representation theory, last-passage percolation, directed landscape, ...)

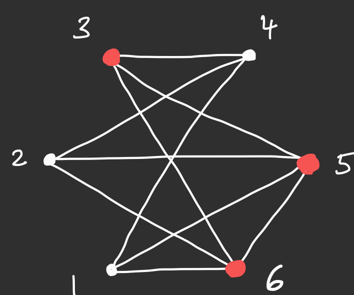
2) LIS of Brownian separable permutons has very nice connections with random trees, random graphs, (oriented) planar maps, SLE/LQG, ...

There are also connections to algebraic geometry (see Kontsevich's metro ticket theorem).



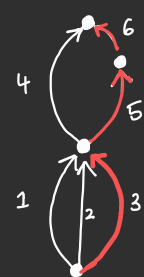
$T_n(p)$ = ⊕/⊖ BINARY TREE

CONNECT ⊕



G_n = BROWNIAN COGRAPHONS

SERIES-PARALLEL COMPOSITION



M_n = SERIES-PARALLEL MAP

$$LIS(\sigma_n) = LPS(T_n) = LCL(G_n) = LDP(M_n)$$

LPS: largest positive subtree; LCL: largest clique; LDP: longest directed path

3) Brownian separable permutons are "critical" for the skew-Brownian permutons, which are universality classes of pattern-avoiding permutations connected to SLE/LQG.

4. PROOF IDEAS

① Existence of an exponent $\alpha(p)$: sub-additivity.

② Randomness helps :

we look at $q(k) = \mathbb{P}(\mathrm{LIS}(T) = k)$

BGW tree with random size.

Looking at what happens at the root, one can write a recursion for $q(k)$.

Problem : the equation is not very tractable...

But it is possible to solve it if we know that

$$q(k) = k^{-\gamma} \underbrace{\varphi(k)}_{\text{slowly varying}} \qquad (*)$$

③ Rémy's algorithm :

Recursive construction $T_n \mapsto T_{n+1}$ by adding a new leaf unif. at random. Let $X_n = \mathrm{LIS}(T_n)$ and $\tau_k = \inf\{n, X_n = k\}$.

Then

$$\mathbb{P}(\mathrm{LIS}(T) \geq k) = \mathbb{P}(X_{|T|} \geq k) = \mathbb{P}(|T| \geq \tau_k) \approx \mathbb{E}\left[\tau_k^{-1/2}\right]$$

size of BGW

To prove $(*)$, we need to prove that

$$\tau_{xk} \approx x^{\tilde{\gamma}} \tau_k .$$

④ Controlling the increments $\tau_{k+1} - \tau_k$:

We apply Rémy's algorithm backwards.

The key identity is

$$X_{n-1} = X_n - \mathbb{1}_{L_n \in \mathcal{L}_n^{max}(T_n)} .$$

where $\mathcal{L}_n^{max}(t)$ = set of leaves in all maximal positive subtrees.

Then we see that

$$\mathcal{L}aw\left(\tau_{k+1} - \tau_k \mid \tau_{k+1}\right) \approx \mathrm{Geom}\left(\frac{\# \mathcal{L}_n^{max}(T_{\tau_{k+1}-1})}{\tau_{k+1}}\right)$$

⑤ A local convergence argument :

The most involved part of the proof is to show a local convergence argument that yields $\# \mathcal{L}_n^{max}(T_n) \approx \alpha X_n$.

This gives

$$\mathcal{L}aw\left(\tau_{k+1} - \tau_k \mid \tau_{k+1}\right) \approx \mathrm{Geom}\left(\frac{\alpha k}{\tau_{k+1}}\right)$$

and with extra work we conclude that

$$\tau_k \approx \mathbb{E}[\tau_k \mid \tau_{xk}] \approx \tau_{xk} \prod_{j=k}^{xk-1} \left(1 - \frac{1}{\alpha j}\right) \approx x^{-\frac{1}{\alpha}} \tau_{xk} .$$