Scaling Limits of critical Fortuin-Kasteleyn planar maps

William Da Silva

joint work with X. Hu, E. Powell, M.D. Wong

universität wien

FWF Austrian Science Fund

OBJECTIVES

I. Fortuin-Kasteleyn maps : INFORMAL RESULTS

II. A queueing model : HAMBURGERS & CHEESEBURGERS

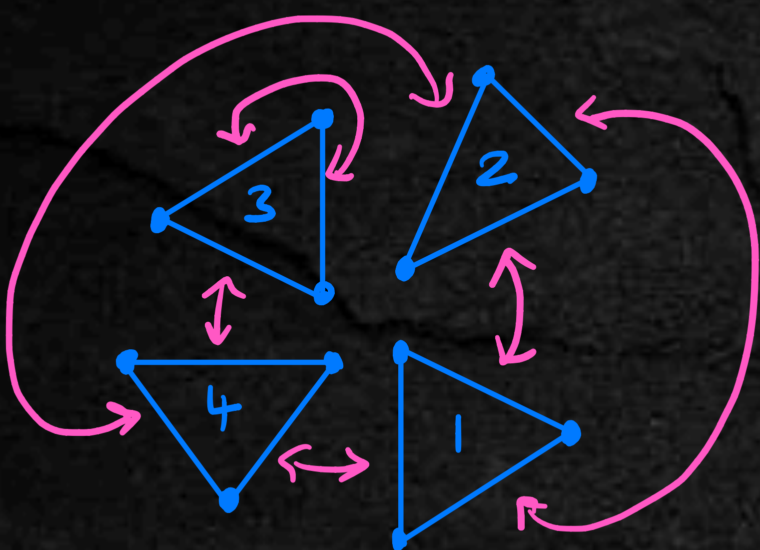
III. A bijective path WORLD I $\longleftrightarrow$ WORLD II

WORLD 1 -

Fortuin - Kasteleyn maps :

Informal Results

$N=4$ triangles

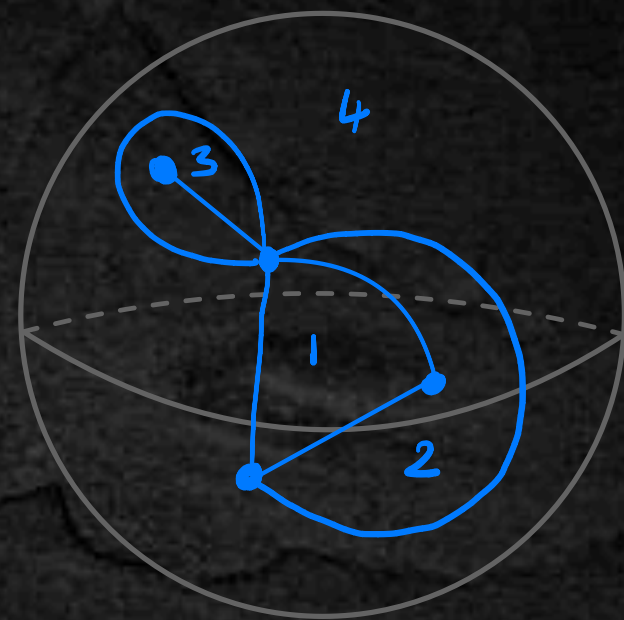


Glue

$\longrightarrow$

at random

sphere $\mathbb{S}^2$



$\approx$ up to homeomorphism

$\mathbb{T}_N$

UNIFORM TRIANGULATION

WHAT DOES IT LOOK LIKE WHEN $N \to \infty$?

$$\boxed{\frac{1}{N^{1/4}} \mathbb{T}_N \xrightarrow{(d)} \mathbb{M}}$$ [as metric spaces]

→ conjectured by Schramm (2006)

→ proved by Le Gall (2011)
Miermont

$\mathbb{M}$ = Brownian map

→ universality:

[Addario-Berry & Albenque], [Bettinelli & Miermont], [Marzouk], etc...

→ techniques: tree bijections

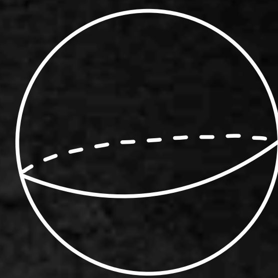
MOTIVATION

Quantum gravity [Polyakov, '81]:

make sense of Random Surfaces

CONSTRUCT A RANDOM METRIC ON THE SPHERE $\mathbb{S}^2$:

γ - Liouville quantum gravity surfaces



$\gamma = 2$ CRITICAL CASE

$\gamma = \sqrt{8/3}$: Brownian map

[Miller - Sheffield]

[Miller - Sheffield - Gwynne]

$\gamma = 0$ [round metric]

How to escape from Brownian Geometry?

TWO WAYS

TO ESCAPE

Maps with "LARGE FACES"

Stable maps

[Curien, Miermont, Riera 2025]

Maps coupled w/ STATISTICAL MECHANICS

e.g. percolation-type models

"gravity coupled to matter"

THE FORTUIN-KASTELEYN MODEL

SAMPLES

- A planar map $\mathbb{m}$ [ROOTED]
- A percolation-like configuration $\mathbb{p}$

  $\rightsquigarrow$ open edges of $\mathbb{m}$

NOTION of dual configuration $\mathbb{p}^*$

$\#cc(\mathbb{p}) = 2$

$\#cc(\mathbb{p}^*) = 3$

LAW:

$$\boxed{\mathbb{P}^{FK(q)}\left[(M,P) = (\mathbb{m}, \mathbb{p})\right] \propto \sqrt{q}^{\,\#cc(\mathbb{p}) + \#cc(\mathbb{p}^*)}}$$

$\mathbb{m}$ = map with N edges, $\mathbb{p} \subset \mathbb{m}$

THE FORTUIN-KASTELEYN MODEL

$$\boxed{\mathbb{P}^{FK(q)}\left[(M,P)=(\mathfrak{m},\mathfrak{p})\right] \propto \sqrt{q}^{\,\#cc(\mathfrak{p})+\#cc(\mathfrak{p}^*)}}$$

$(\mathfrak{m}, \mathfrak{p})$

$\mathfrak{m}$ = map with N edges, $\mathfrak{p} \subset \mathfrak{m}$

⇝ Given M, $P \sim$ FK(q)-percolation on M

⇝ $q=1$ bond percolation

$q=0$ spanning trees

⇝ connections with

Ising, Potts, loop-$O(n)$

models

⇝ local limit (M_∞, P_∞) $[N \to \infty]$

INFORMAL RESULT

2D "Random Walk" encoding of (M_∞, P_∞) : $(S_k, D_k)_{k \in \mathbb{Z}}$.

THEOREM [Sheffield 2011]

We have $\frac{1}{\sqrt{k}} \left(S_{\lfloor kt \rfloor}, D_{\lfloor kt \rfloor} \right)_{t \in \mathbb{R}} \xrightarrow{d} \left(B^{(1)}_t , B^{(2)}_{\alpha(q) t} \right)_{t \in \mathbb{R}}$ [independent Brownian motions]

$\alpha(q)$ is explicit , $\alpha(q) = \begin{cases} > 0 & \text{if } q < 4 \\ = 0 & \text{if } q \geq 4 \end{cases}$

⤳ [PEANOSPHERE] CONVERGENCE OF FK(q) PLANAR MAPS WITH $q < 4$

TOWARDS

γ - LIOUVILLE QUANTUM GRAVITY WITH $\gamma \in [\sqrt{2}, 2)$ s.t. $q = 2 + 2\cos\left(\frac{\pi \gamma^2}{2}\right)$

[Nienhuis]

INFORMAL RESULT

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THEOREM [DS-Hu-Powell-Wong]

At $q=4$, we have that $\left(\frac{1}{\sqrt{k}} S_{\lfloor kt\rfloor}, \frac{\log k}{2\pi\sqrt{k}} D_{\lfloor kt\rfloor}\right)_{t\in\mathbb{R}} \xrightarrow{d} \left(B^{(1)}_t, B^{(2)}_t\right)_{t\in\mathbb{R}}$

$\rightsquigarrow$ CONVERGENCE TO CRITICAL [$\gamma=2$] LQG

WORLD II -

A Queuing Model :

Hamburgers & Cheeseburgers

THE HAMBURGER - CHEESEBURGER MODEL

- Alphabet $\Theta = \{h, c, H, C, F\}$

- Word w in Θ $\Longleftrightarrow$ "day in the life of a restaurant".

EXAMPLE

$w = hcHHF$

hamburger

$t = 1$

THE HAMBURGER-CHEESEBURGER MODEL

- Alphabet $\Theta = \{h, c, H, C, F\}$
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EXAMPLE

$w = hcHHF$

cheeseburger

$t = 2$

THE HAMBURGER-CHEESEBURGER MODEL

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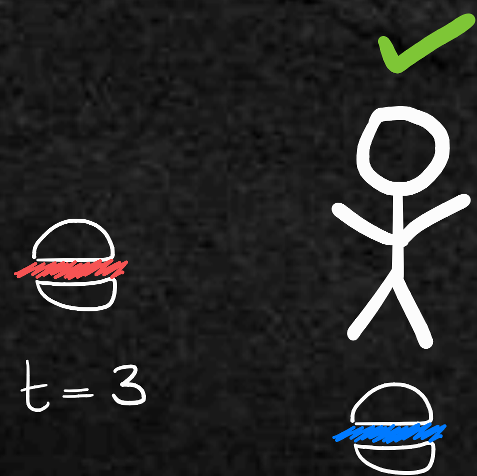

$t = 3$

hamburger order

THE HAMBURGER - CHEESEBURGER MODEL

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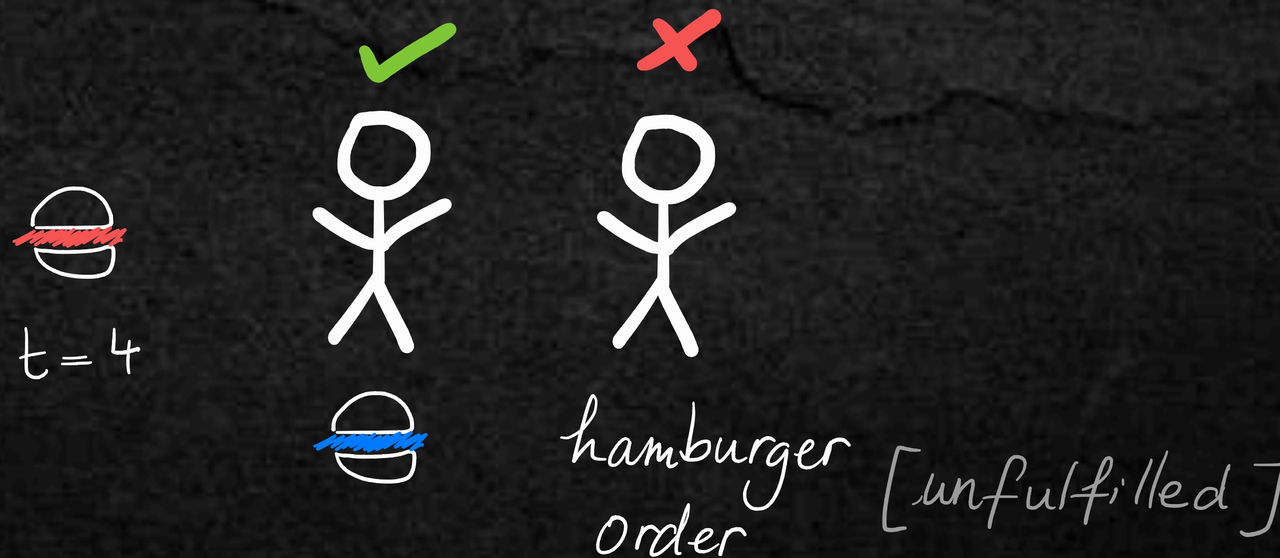


THE HAMBURGER-CHEESEBURGER MODEL

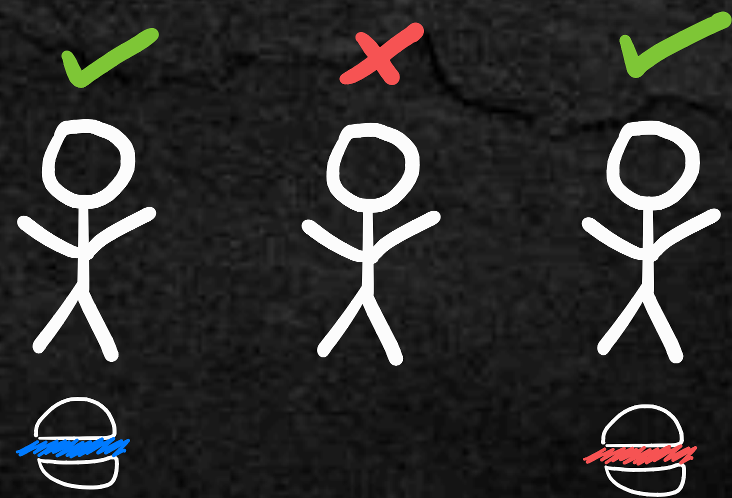
- Alphabet $\Theta = \{h, c, H, C, F\}$
- Word w in Θ $\Longleftrightarrow$ "day in the life of a restaurant".

EXAMPLE

$w = hcHHF$

ϕ

$t = 5$



fresh order

THE HAMBURGER-CHEESEBURGER MODEL

- Alphabet $\Theta = \{h, c, H, C, F\}$

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EXAMPLE

$w = hcHHF$

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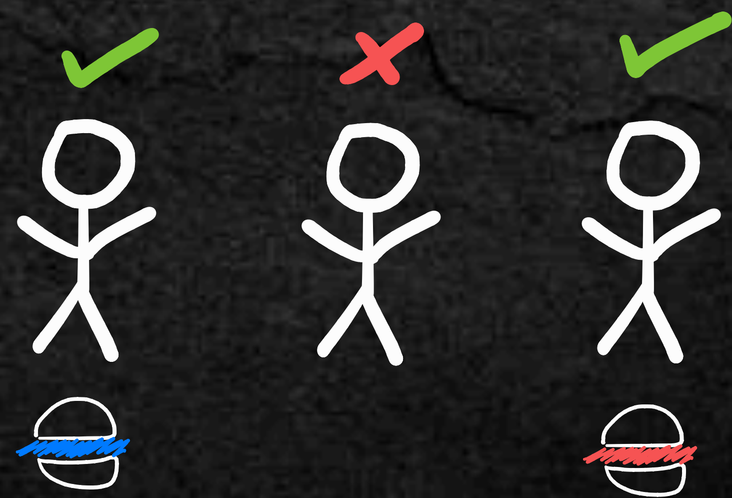


EVOLUTION OF BURGER STACK

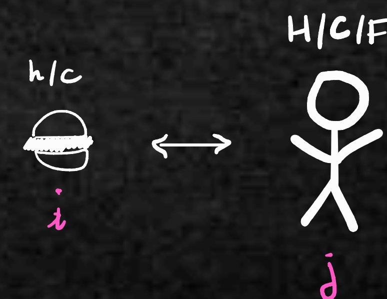
THE HAMBURGER-CHEESEBURGER MODEL
WITH RANDOMNESS

Let $p \in [0,1]$ and take

$$\mathbb{P}(h) = \mathbb{P}(c) = \frac{1}{4} \qquad \mathbb{P}(H) = \mathbb{P}(C) = \frac{1-p}{4} \qquad \mathbb{P}(F) = \frac{p}{2}$$

RANDOM WORD $W = \underbrace{\cdots X(-1)\, X(0)\, X(1) \cdots}_{\text{Bi-infinite iid sequence}}$

$i \in \mathbb{Z}$ $j \in \mathbb{Z}$ are a match:



(if $i<j$)

PROPOSITION [Sheffield 2011]

Almost surely, every $i \in \mathbb{Z}$ has a match $\varphi(i)$ in W.

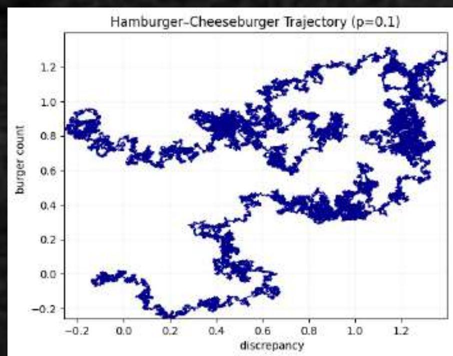
BURGER COUNT & DISCREPANCY

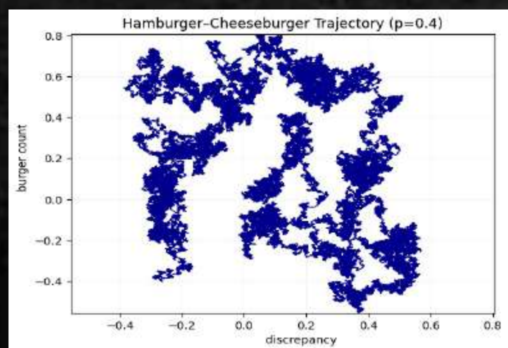
NOTATION

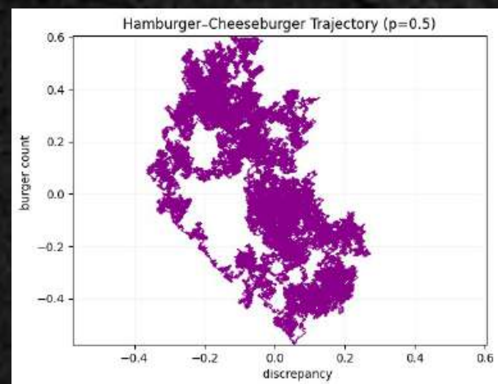
- hamburger counts $\mathcal{H}(i,j) :=$ "$\#h - \#H - \#[F \leftrightarrow h]$"
  cheeseburger counts $\mathcal{C}(i,j) :=$ "$\#c - \#C - \#[F \leftrightarrow c]$"

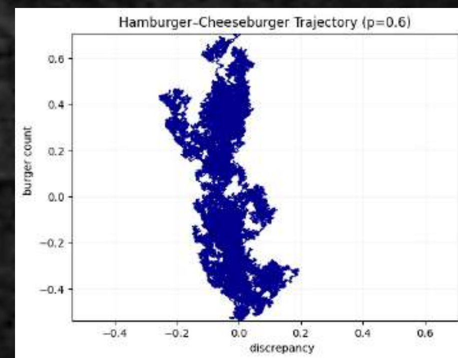
- BURGER COUNT $S(i,j) := \mathcal{H}(i,j) + \mathcal{C}(i,j)$
  DISCREPANCY $\mathcal{D}(i,j) := \mathcal{H}(i,j) - \mathcal{C}(i,j)$

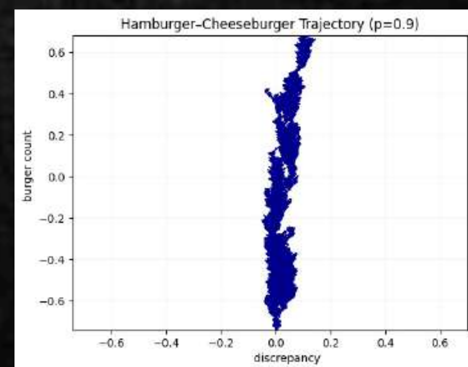
SIMULATIONS OF $\frac{1}{\sqrt{n}}\big(\mathcal{D}(1,n), S(1,n)\big)$











MAIN RESULT

Recall $W = \cdots X(-1)\, X(0)\, X(1) \cdots$

$(S_t, \mathcal{D}_t)_{t\in\mathbb{R}}$ $\quad S_0 = \mathcal{D}_0 = 0 \quad$ $(S_t, \mathcal{D}_t) = (S(1,t), \mathcal{D}(1,t)) \quad t>0$

$(S_t, \mathcal{D}_t) = (-S(1,t), -\mathcal{D}(1,t)) \quad t<0$

THEOREM [Sheffield 2011]

Let $\alpha(p) = \max(0, 1-2p)$

We have $\frac{1}{\sqrt{k}}(S_{\lfloor kt \rfloor}, \mathcal{D}_{\lfloor kt \rfloor})_{t\in\mathbb{R}} \xrightarrow{d} (B_t^{(1)}, B_{\alpha(p)t}^{(2)})_{t\in\mathbb{R}}$ with $B^{(1)} \perp\!\!\!\perp B^{(2)}$

$\rightsquigarrow$ PHASE TRANSITION at $p = \frac{1}{2}$

THEOREM [DS-Hu-Powell-Wong]

At $p = \frac{1}{2}$ we have: $\left(\frac{1}{\sqrt{k}} S_{\lfloor kt \rfloor}, \frac{\log(k)}{2\pi\sqrt{k}} \mathcal{D}_{\lfloor kt \rfloor}\right)_{t\in\mathbb{R}} \xrightarrow{d} (B_t^{(1)}, B_t^{(2)})_{t\in\mathbb{R}}$

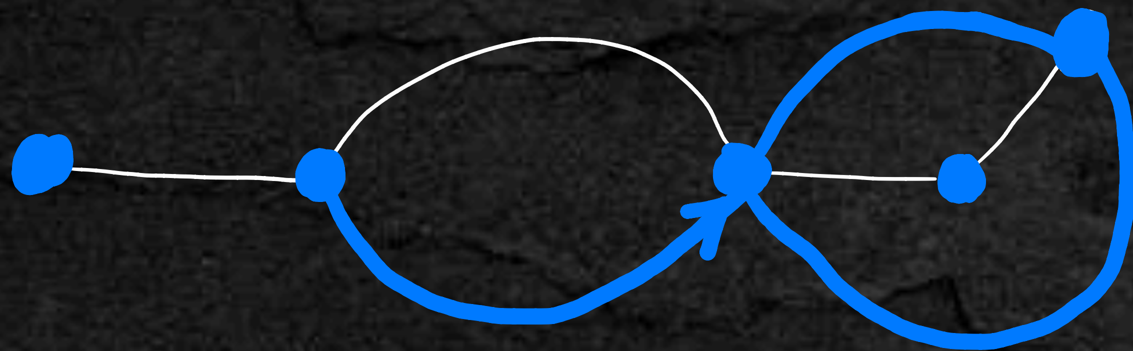
III -

A Bijective Path between World I & World II

THE HAMBURGER - CHEESEBURGER BIJECTION

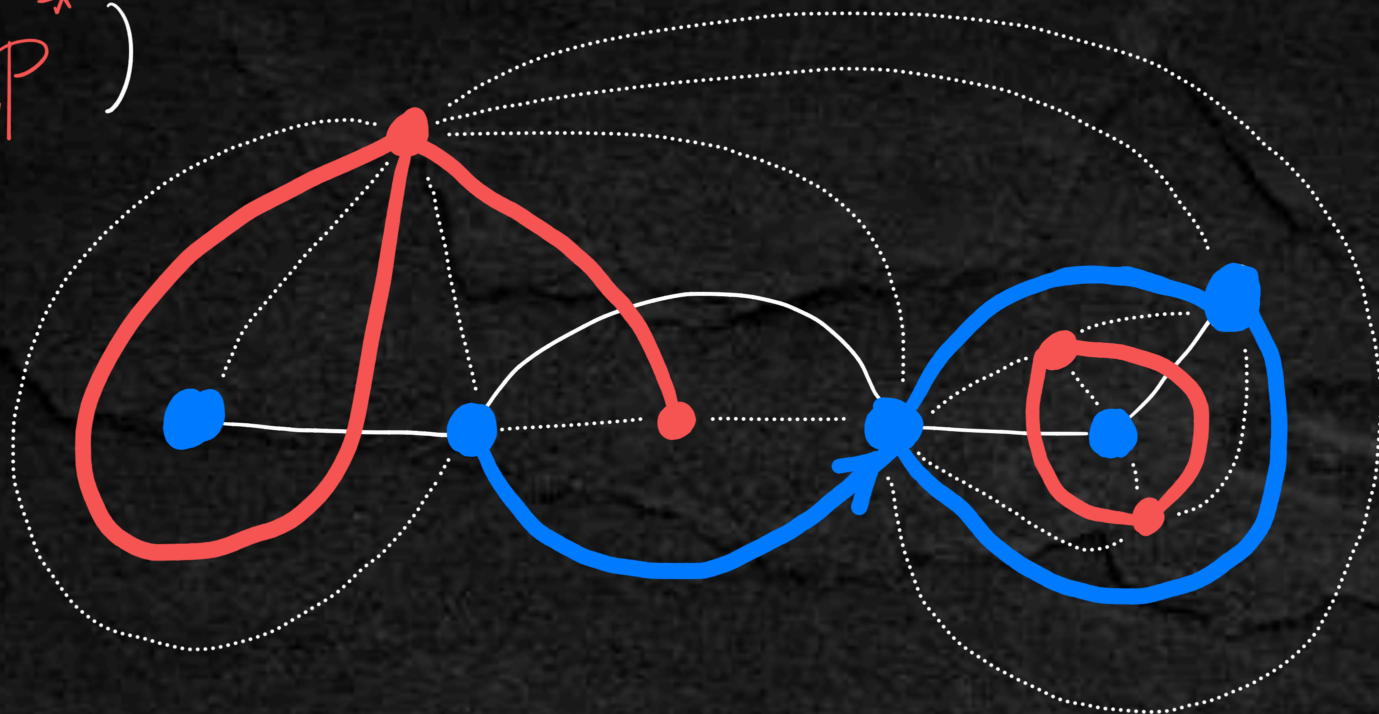
$(\mathfrak{m}, \mathbb{P})$

[Mullin]
[Bernardi]
[Sheffield]



THE HAMBURGER - CHEESEBURGER BIJECTION

$(\mathrm{m}, \mathbb{P}, \mathbb{P}^*)$



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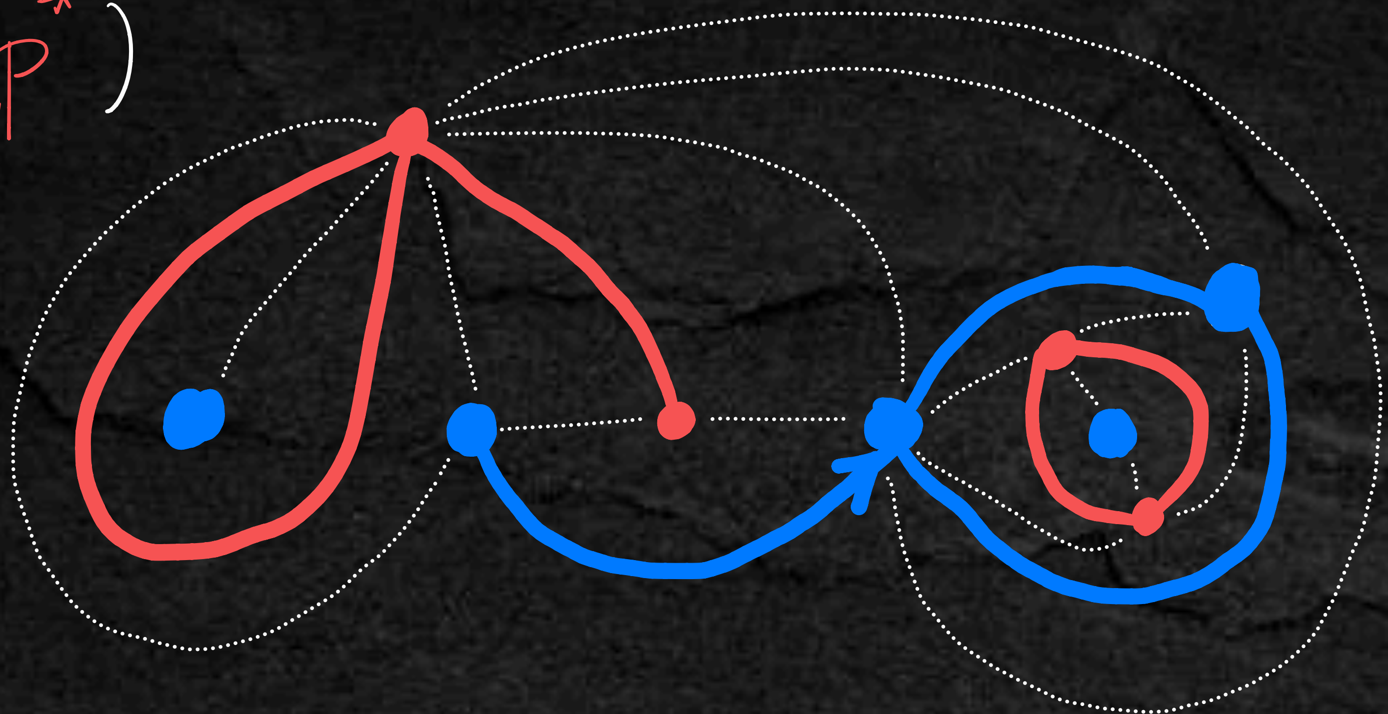

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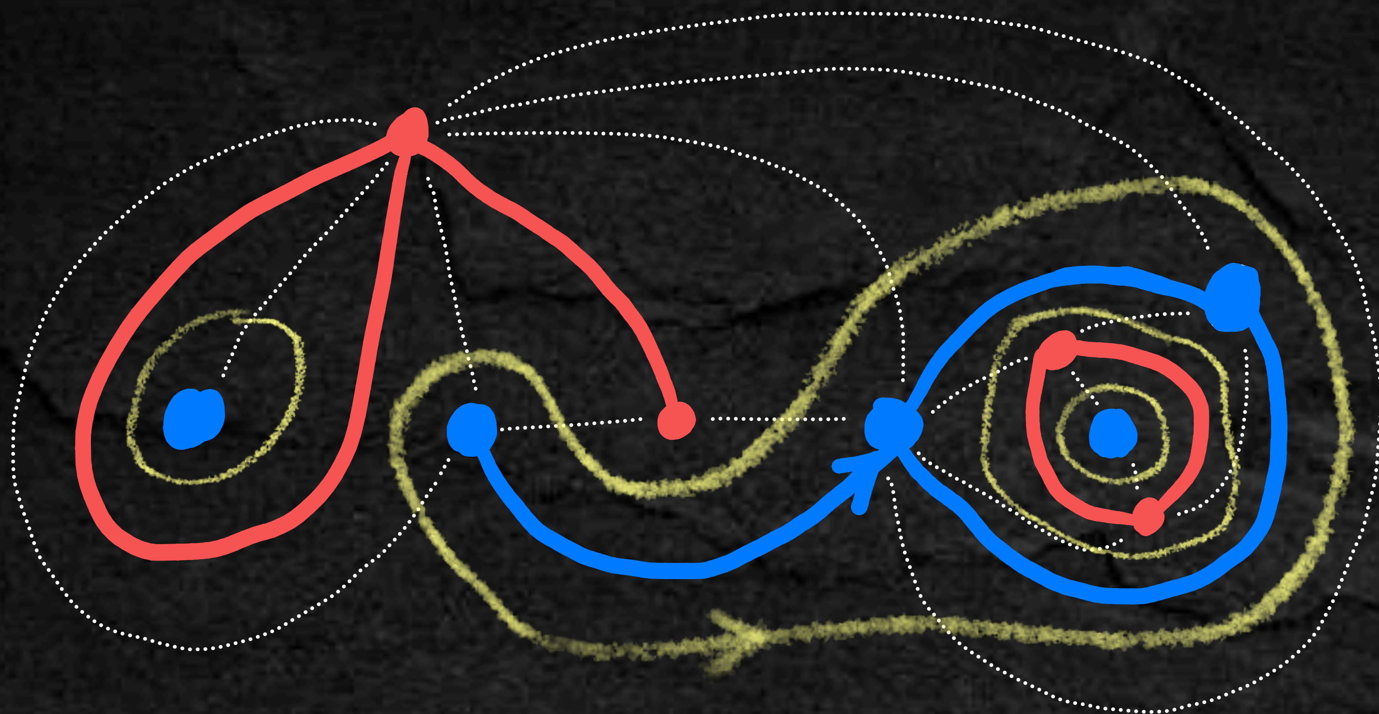

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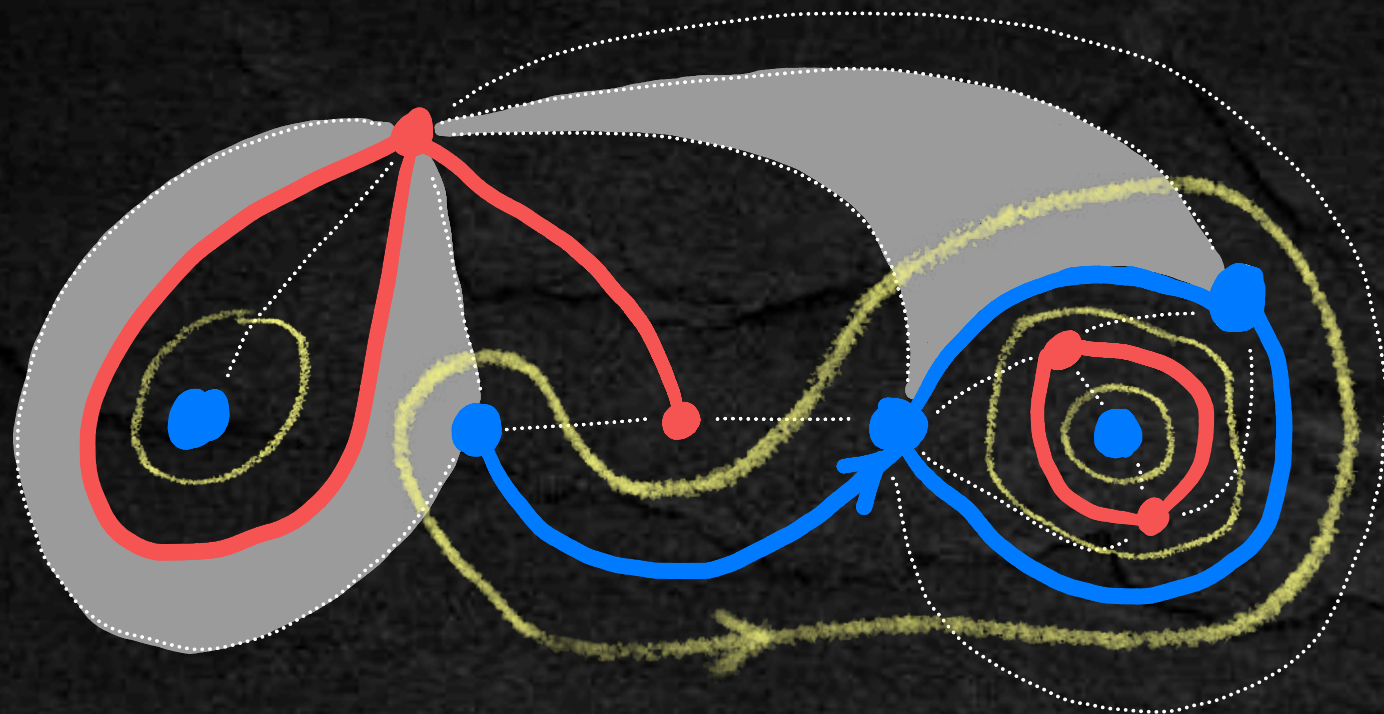

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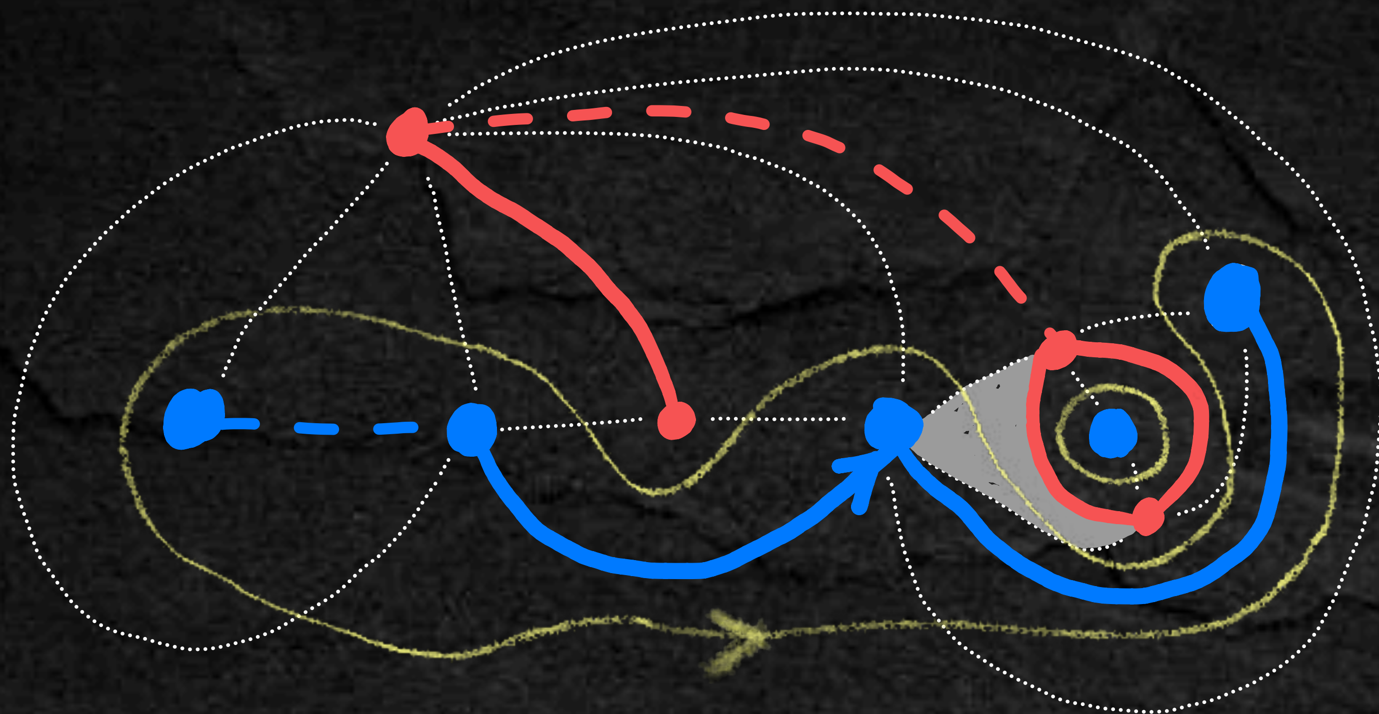

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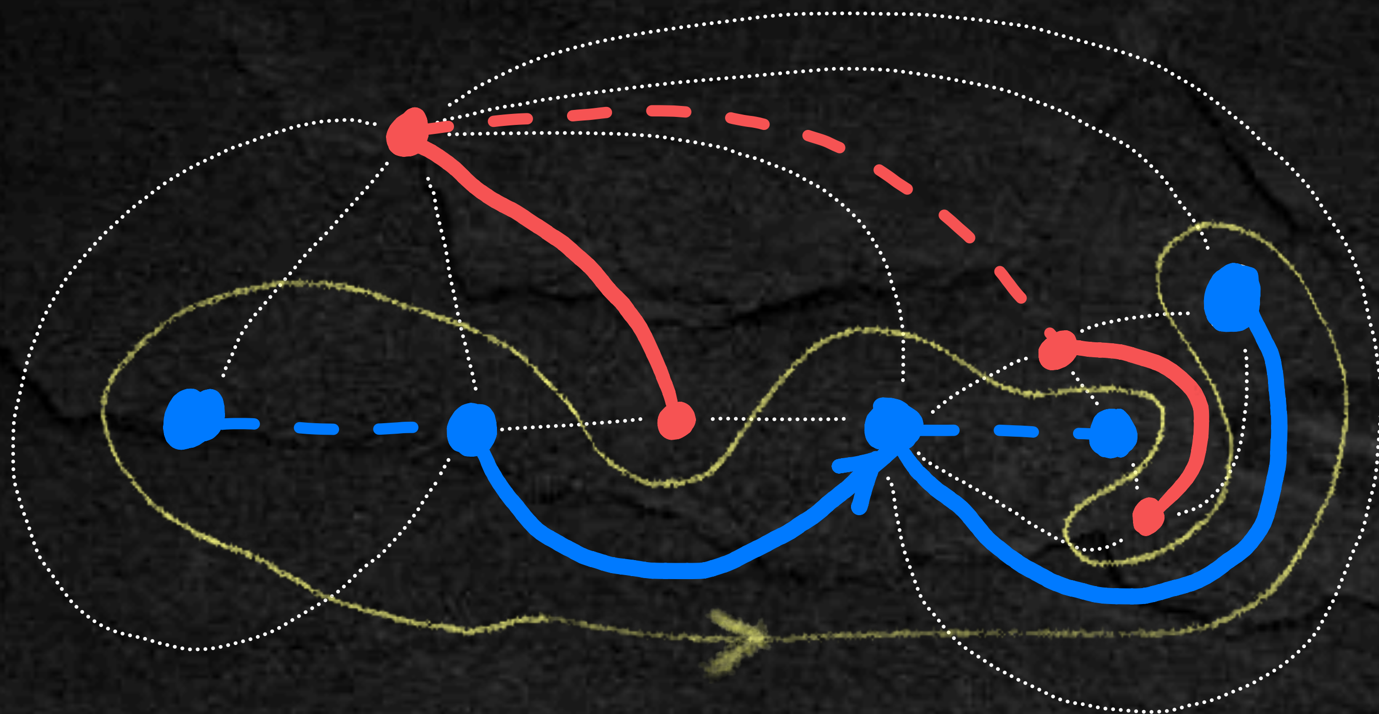

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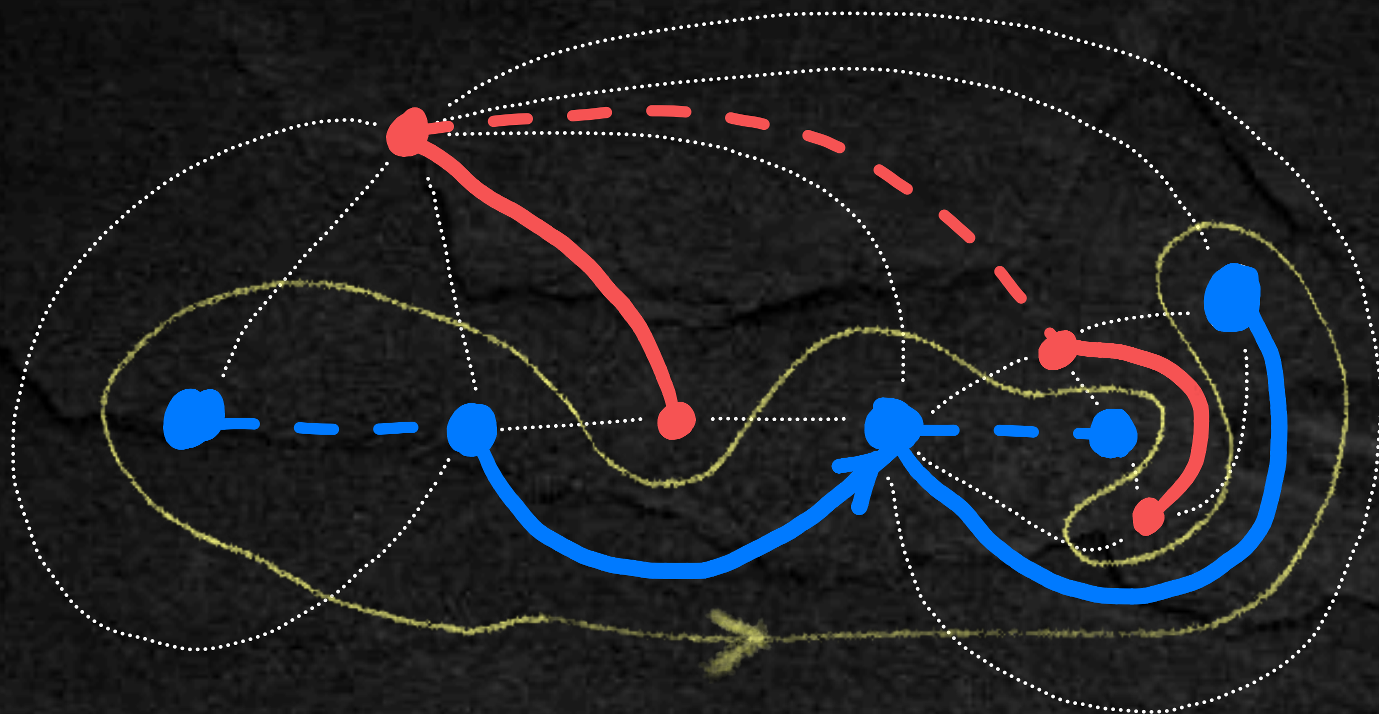
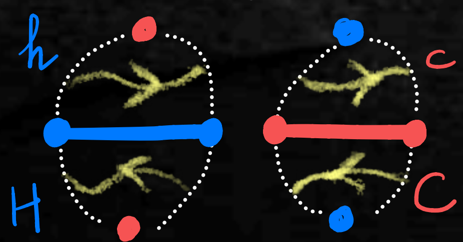


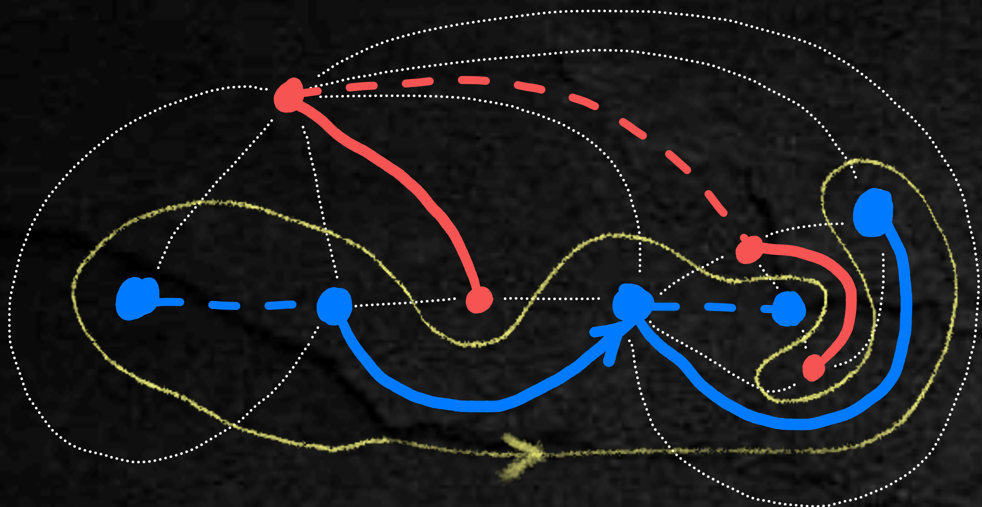
Triangles $\longrightarrow$ Word



$$W(\mathbb{m}, \mathbb{p}) = hhccHhCFFcHChF$$

in $\Theta := \{h, c, H, C, F\}$

THE HAMBURGER - CHEESEBURGER BIJECTION



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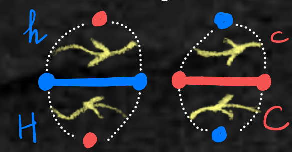


$w(\mathbb{m}, \mathbb{p}) = hhccHhCFFcHChF$

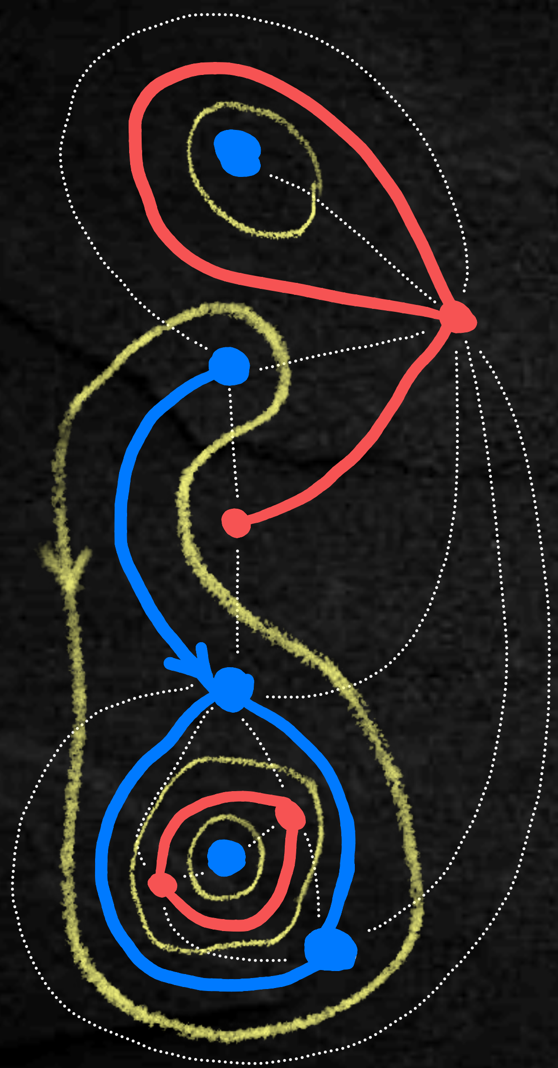
in $\Theta := \{h, c, H, C, F\}$

FACT

$\mathbb{m}$: Infinite FK(q) maps $\longrightarrow$ p- (iid) hamburger-cheeseburger model is a bijection

with $\sqrt{q} = \frac{2p}{1-p}$

A FEW GEOMETRIC PROPERTIES

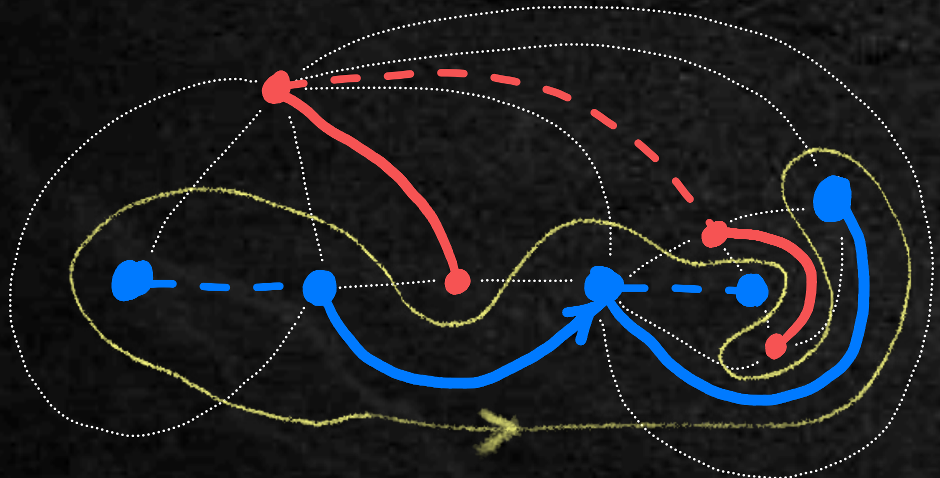


- Loops $\longleftrightarrow$ F symbols
- Open edges $\longleftrightarrow$ "hamburgers"
  blue
- Dual edges $\longleftrightarrow$ "cheeseburgers"
  red

A FEW GEOMETRIC PROPERTIES

"MATING OF TREES"

Recall $\mathcal{H}(i,j) :=$ "$\#h - \#H - \#[F \leftrightarrow h]$"
$\mathcal{C}(i,j) :=$ "$\#c - \#C - \#[F \leftrightarrow c)]$"



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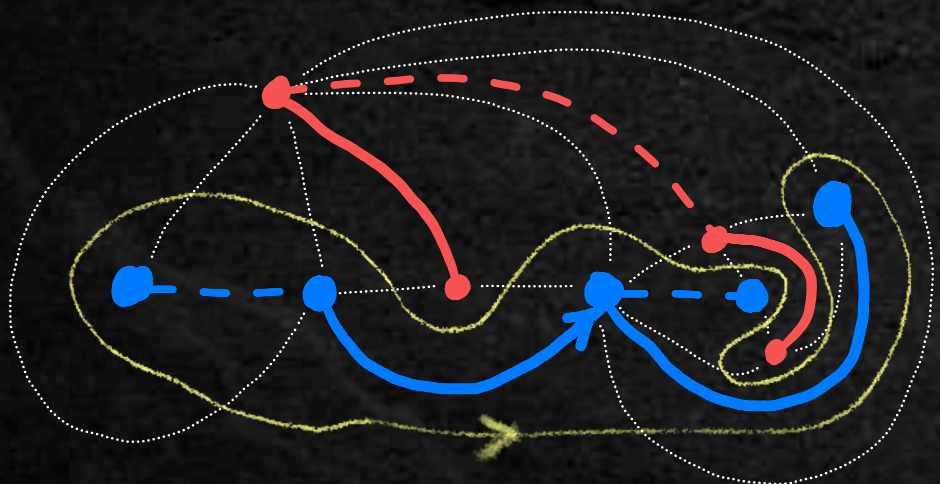

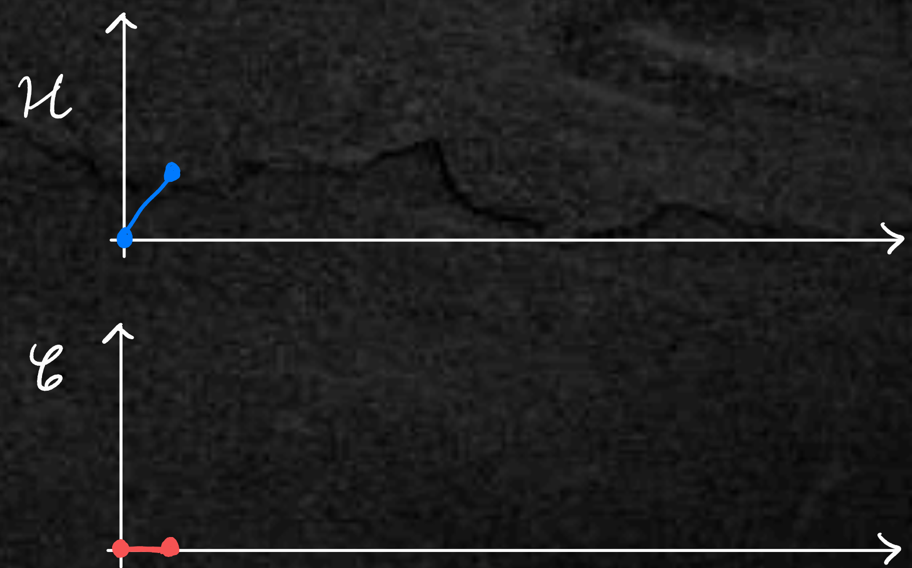

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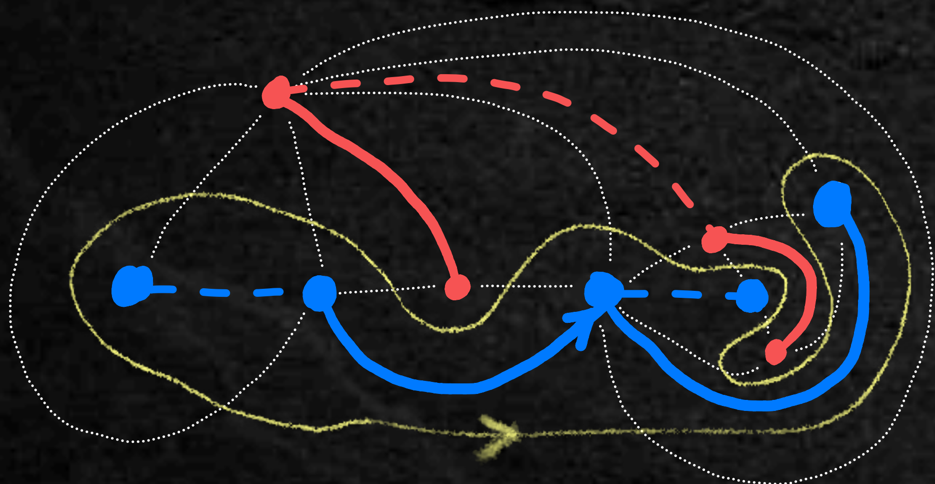

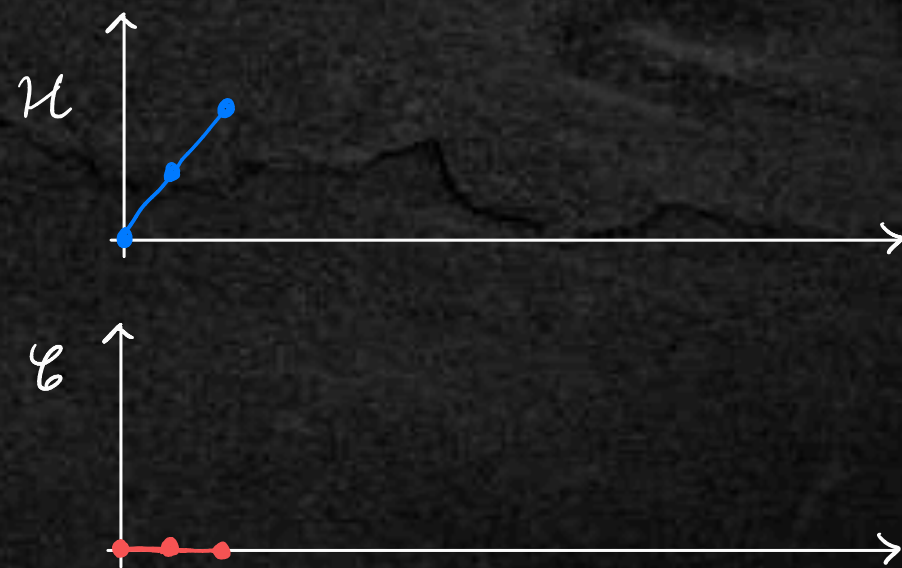

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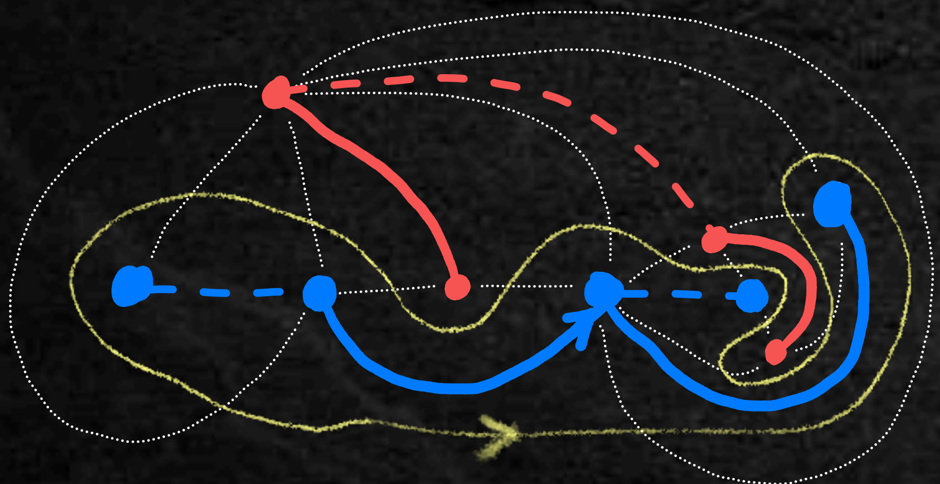

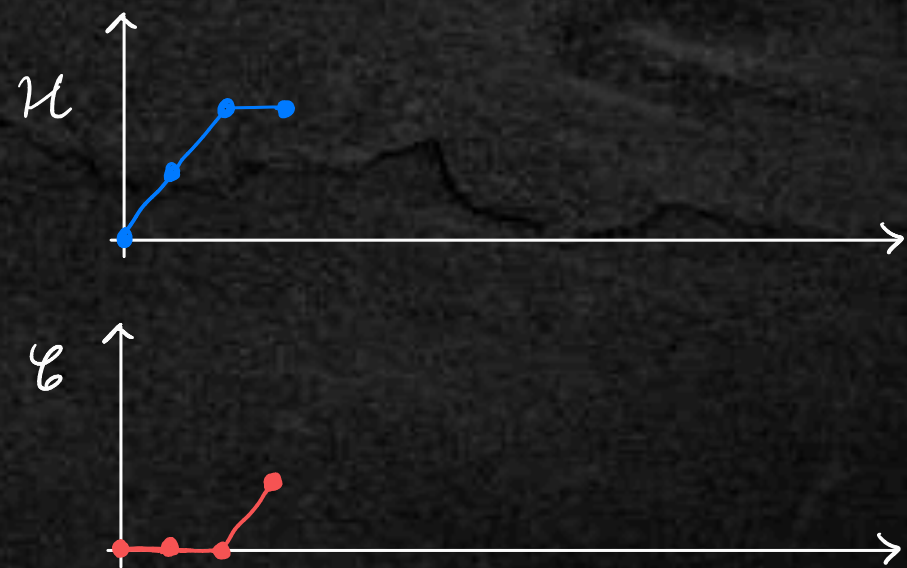

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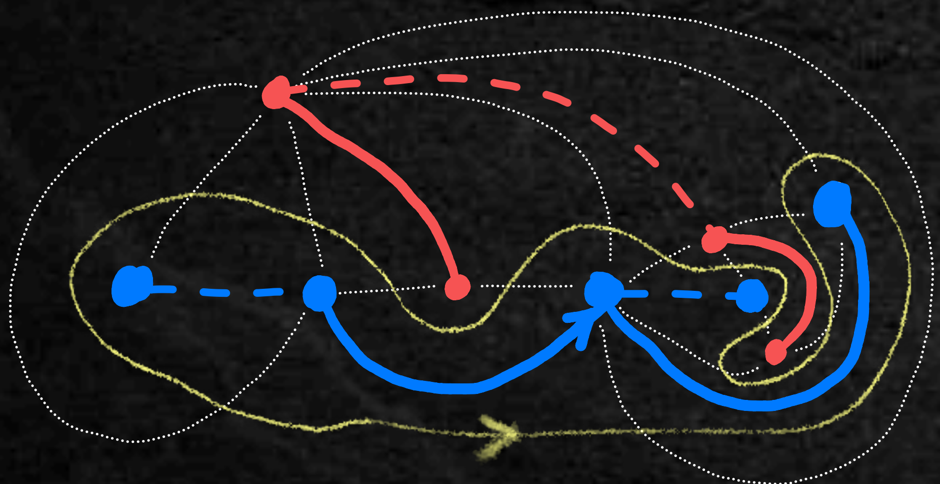
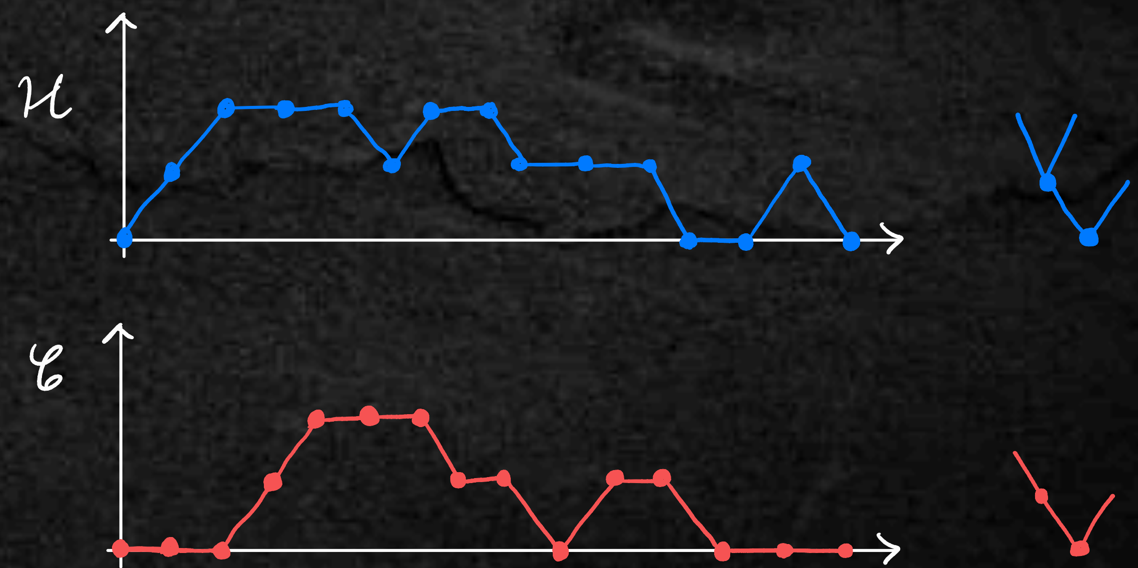




A WORD ABOUT THE PROOF

$\boxed{q \in (0,4)}$

- SHEFFIELD'S PROOF:

  martingale techniques

- CRITICAL EXPONENTS:

  [Berestycki – Laslier – Ray '17]

  [Gwynne – Mao – Sun '19]

$$\mathbb{P}(|\partial \mathfrak{C}(0)| = \ell) = \ell^{-\alpha(q)+o(1)}$$

$\boxed{q = 4}$

- CORRESPONDENCE

  FULLY PACKED O(2) TRIANGULATIONS

  + gasket decomposition [Borot - Bouttier - Guitter '11]

- EXACT SOLVABILITY

  * partition function F_ℓ

    [Gaudin – Kostov '89]

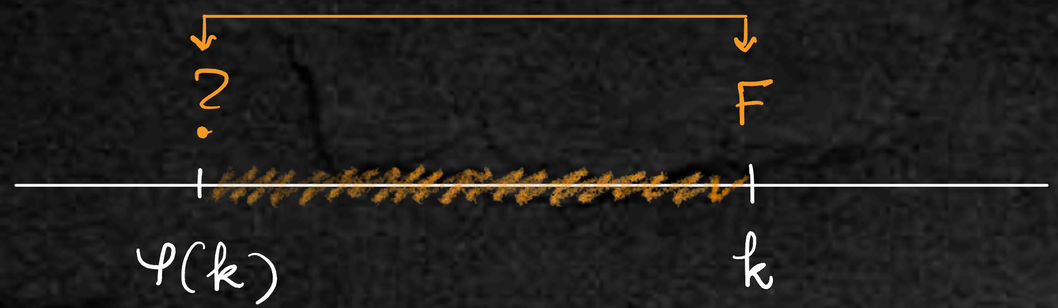
  * exact expressions

$$\mathbb{P}(|\partial \mathfrak{C}(0)| = \ell) = g(\ell) \cdot (2\sqrt{2})^{-\ell} F_\ell$$

j.w. N. Berestycki

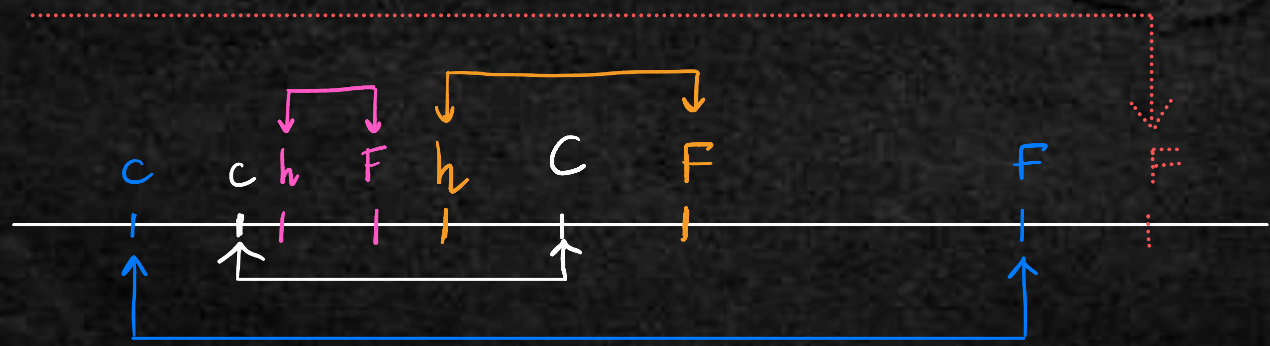
CHALLENGES

Discrepancy isn't Markov



$\mathcal{D}(k) = ?$

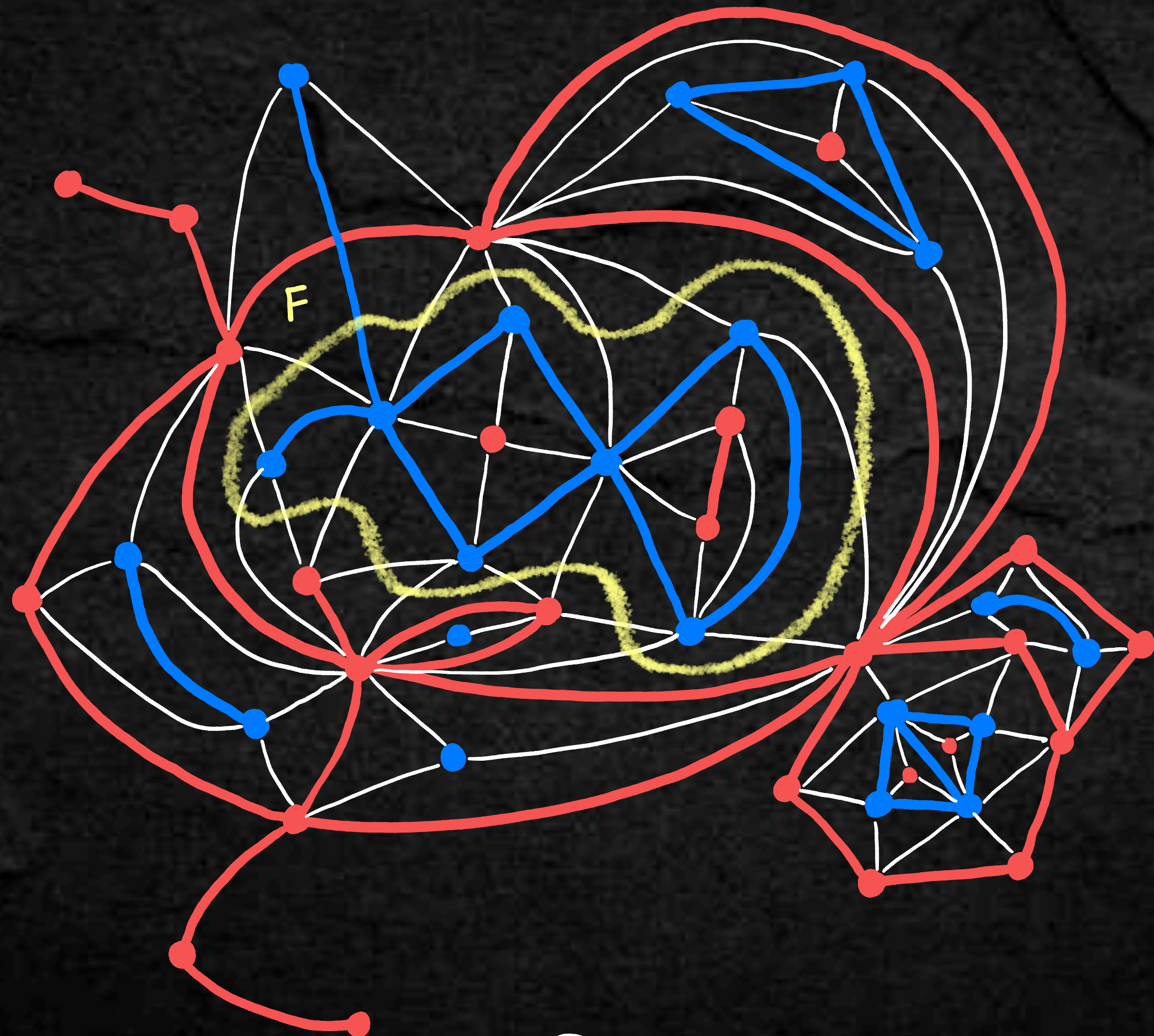
even worse :



Important estimate : $\#\{ i \le n \mid X(i) = F \ \& \ \varphi(i) < 0 \}$ is negligible

CHALLENGES

Correspondence "MAP ⟷ BURGERS" is VERY SUBTLE



CHALLENGES

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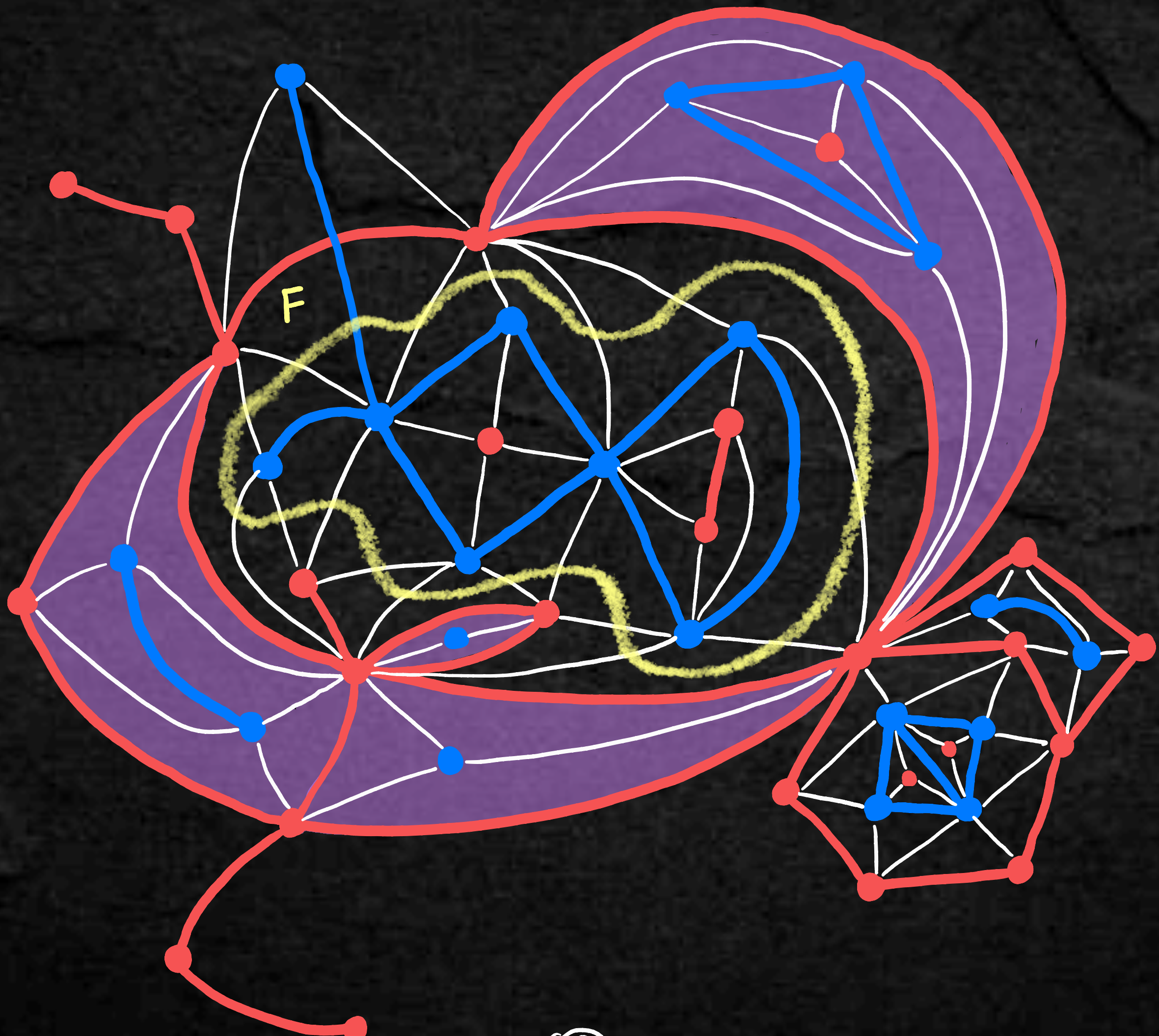

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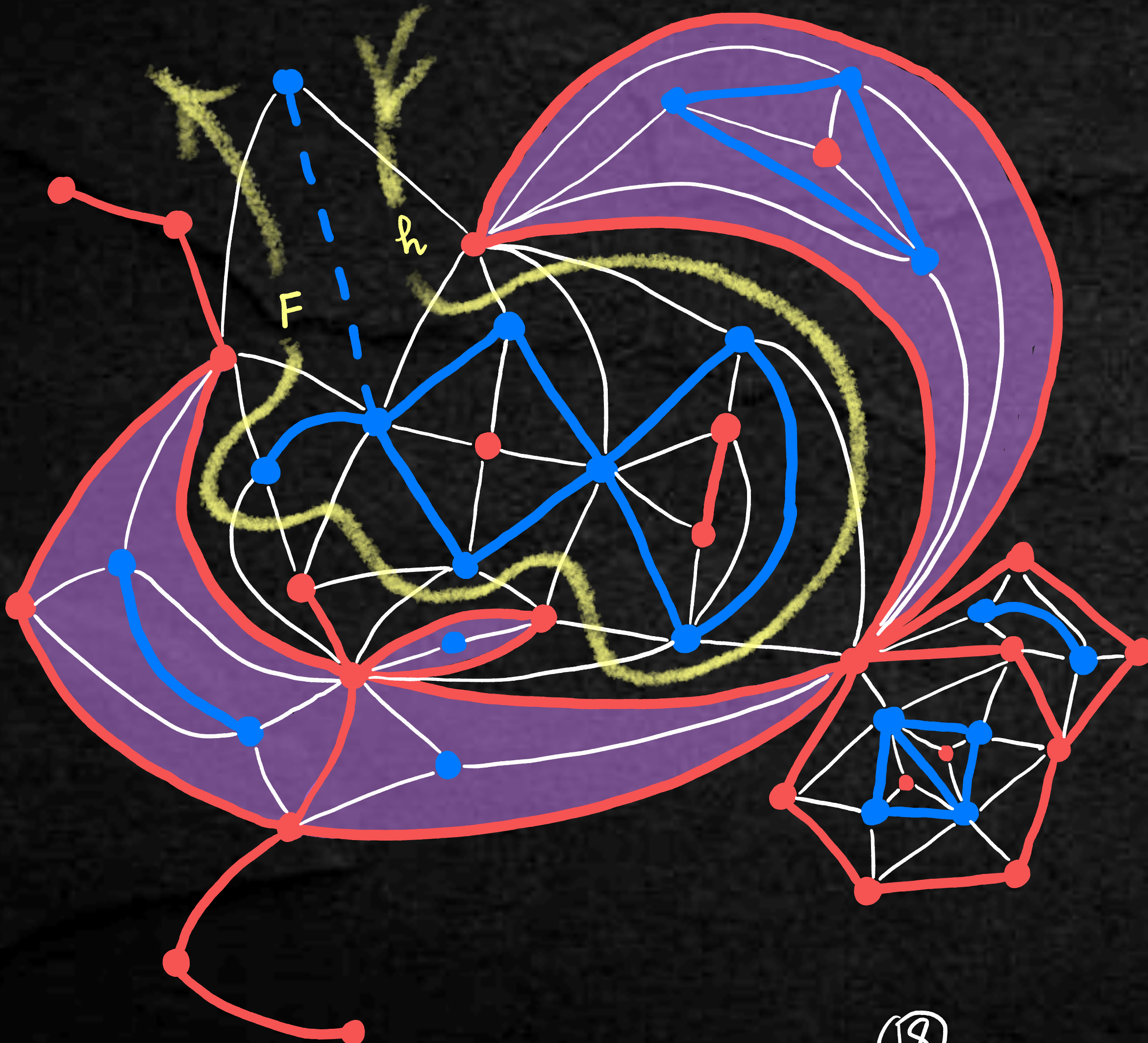
Correspondence "MAP $\longleftrightarrow$ BURGERS" is VERY SUBTLE



h   h...F   c...C   h...F   F

CHALLENGES

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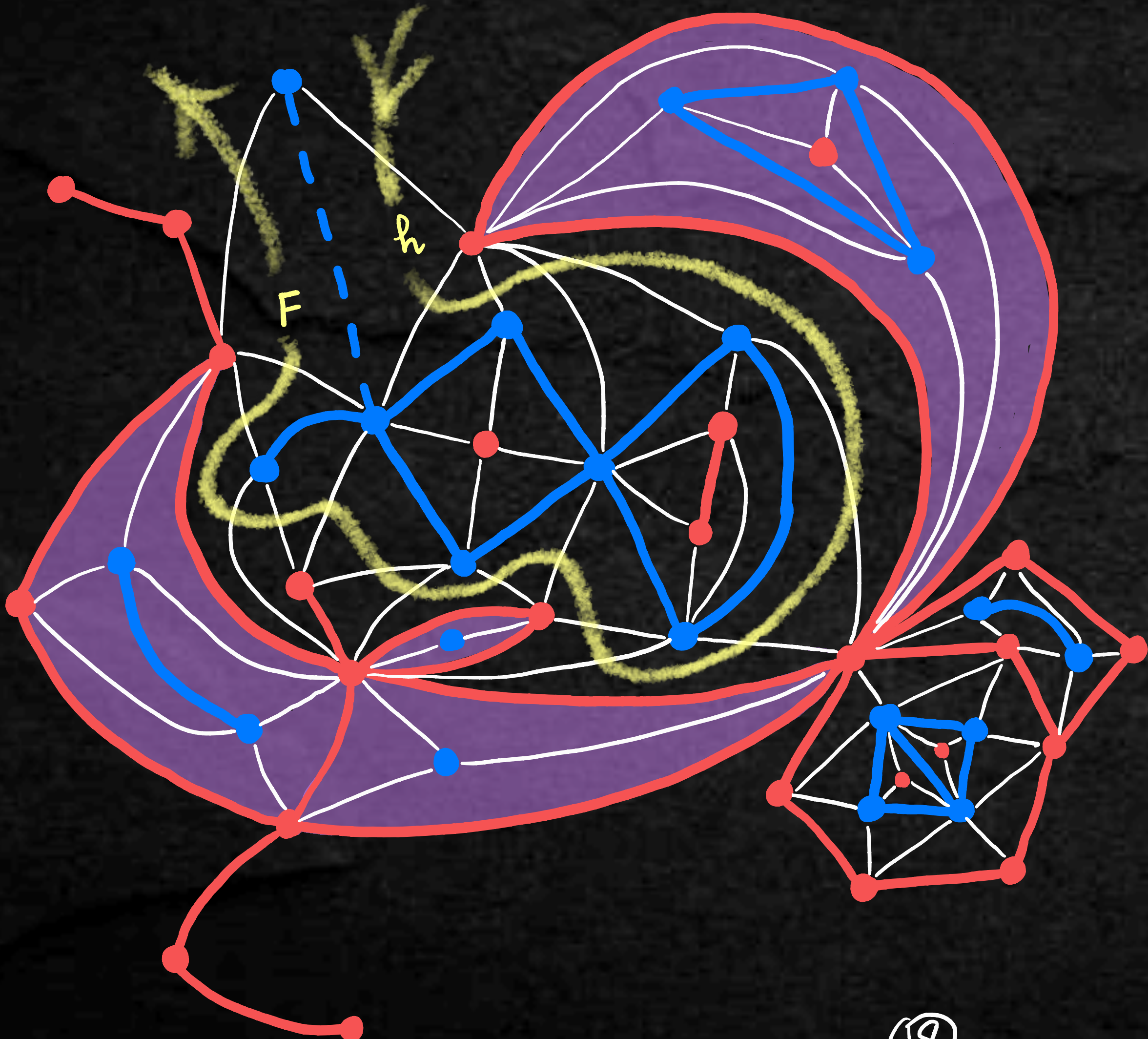

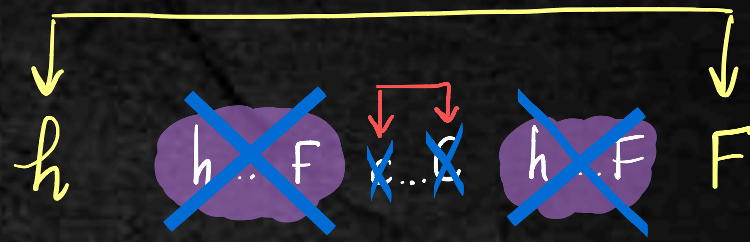

$\rightsquigarrow$ SKELETON

PARTITION FUNCTION $\longleftrightarrow$ $\mathcal{F}_\ell$

CHALLENGES

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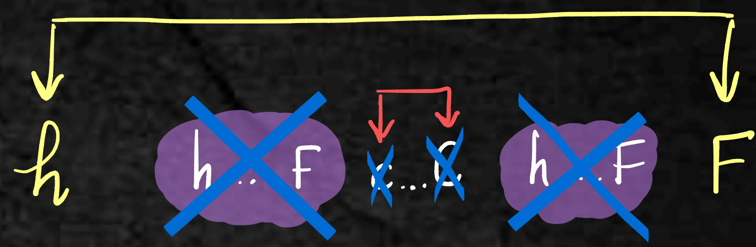
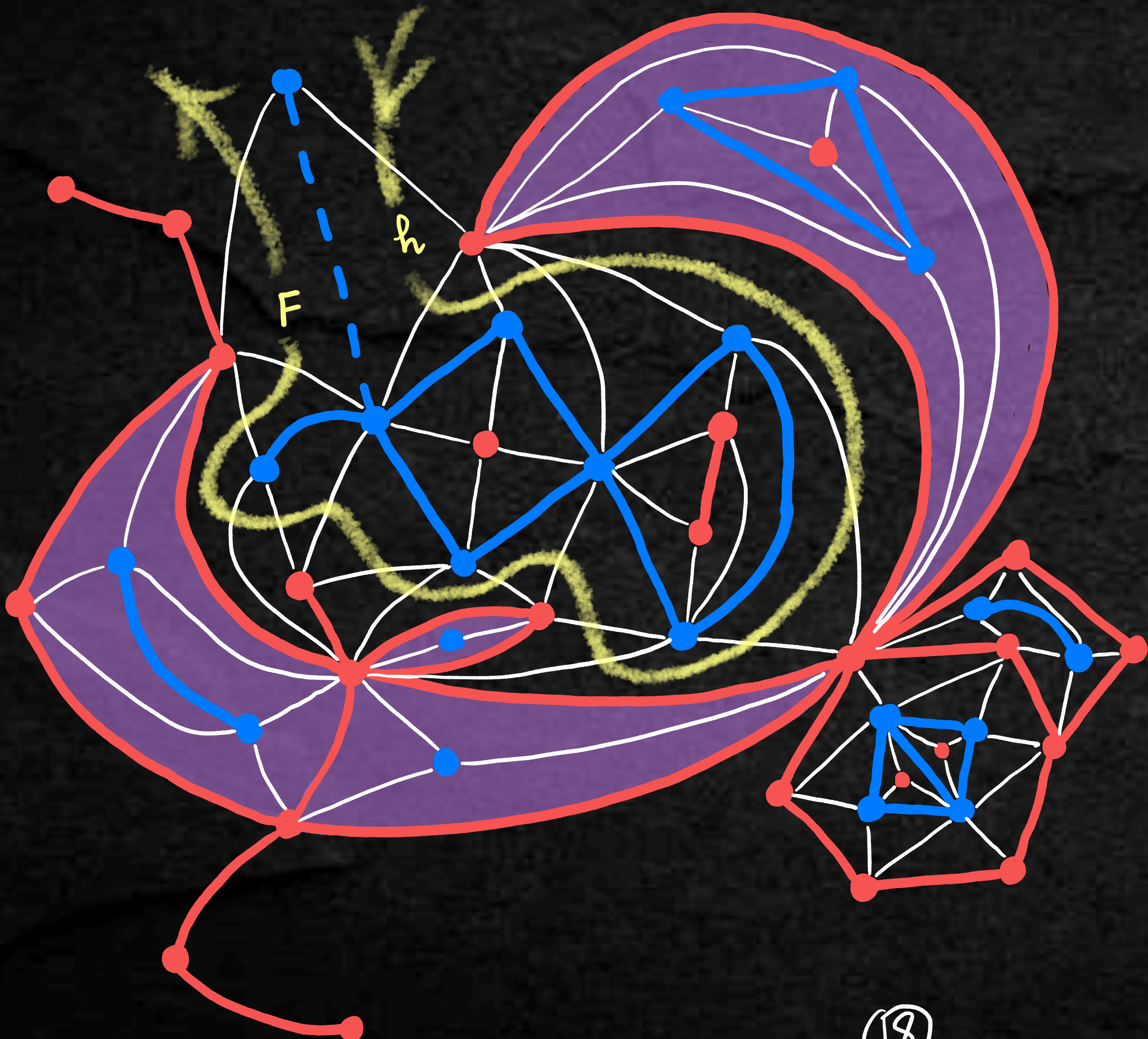




$\rightsquigarrow$ SKELETON

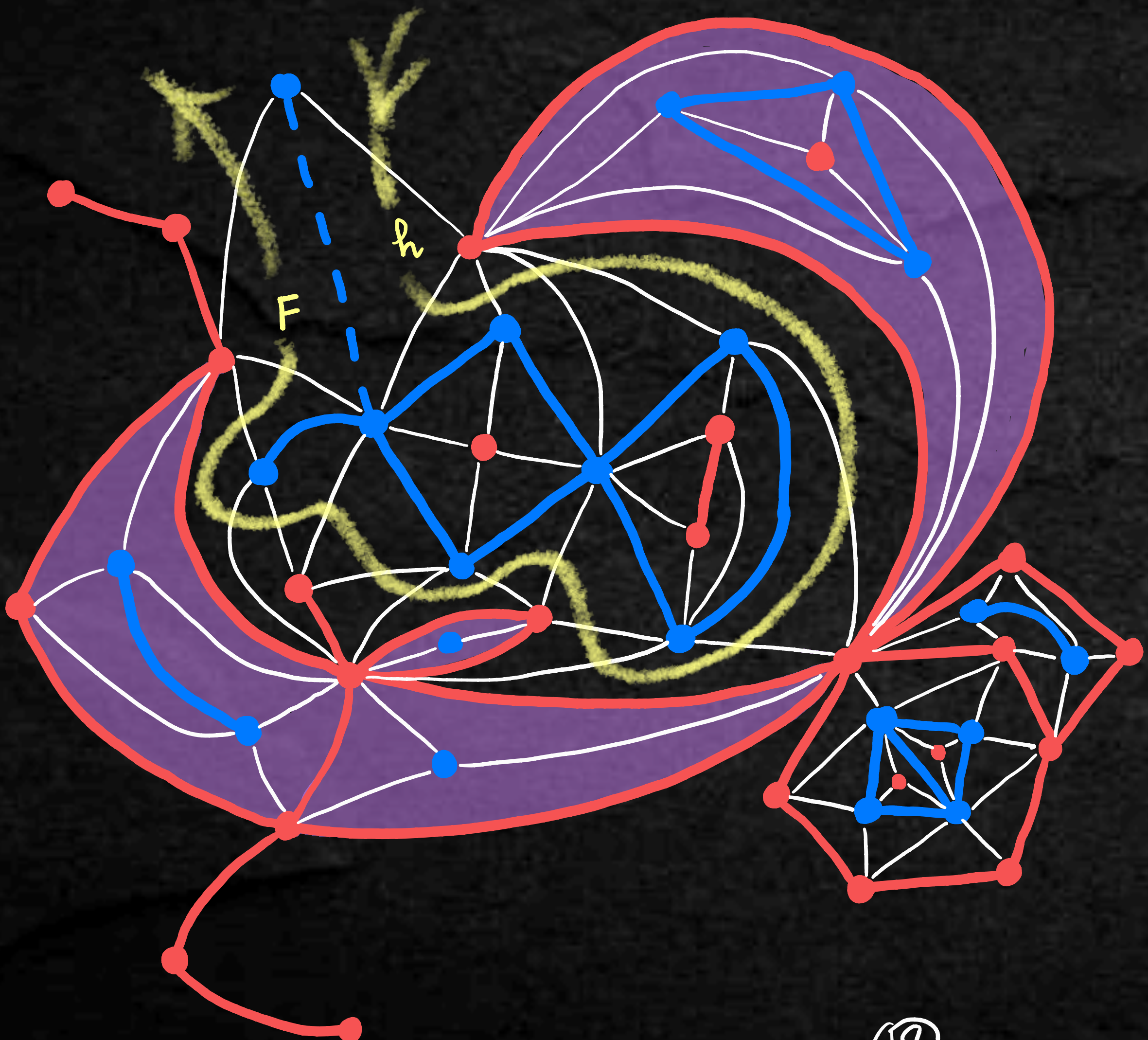
PARTITION FUNCTION $\longleftrightarrow$ $\mathcal{F}_\ell$

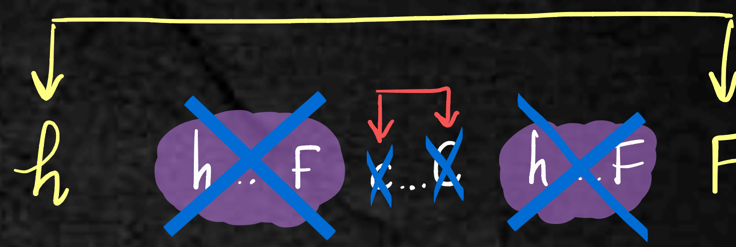
$\rightsquigarrow$ EXPLORATION INTO THE PAST

$\mathcal{F}_\ell \Rightarrow$ Random walk estimates

CHALLENGES

Correspondence "MAP $\longleftrightarrow$ BURGERS" is VERY SUBTLE





$\rightsquigarrow$ SKELETON

PARTITION FUNCTION $\longleftrightarrow$ $\mathcal{F}_\ell$

$\rightsquigarrow$ EXPLORATION INTO THE PAST

$\mathcal{F}_\ell \Longrightarrow$ Random walk estimates

$\rightsquigarrow$ FUTURE ?